Oligopolistic Competition, Time-Inconsistency and the Firm’s Valuation*

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Abstract

We consider a representative agent economy with one consumption good, one capital good and \( n \) non-price taking value maximizing firms that choose the level of investment and production over the finite horizon. As noted in the literature, the firm’s ability to affect the state prices in the economy leads to the time-inconsistency of its objective function. We characterize the effect of oligopolistic competition on the production and investment both when the firms can pre-commit to their strategies and when they use time-consistent strategies. We show that the inability to pre-commit induces overproduction beyond the competitive level leading to lower valuation of the firms. Moreover, the inability to pre-commit completely destroys the value of the firm and leads to the disinvestment of capital as the decision-making interval shrinks to zero. We also derive conditions under which the accumulation of capital is optimal in a symmetric equilibrium in this economy. We show that for a small decision-making interval, holding other parameters fixed, the firms choose to increase their stock of capital good only when the competition in the industry is substantially high.

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1. Introduction

The importance of imperfect competition has always been recognized not only by economists but also by policymakers. One of the major concerns has always been the impact of monopolistic and oligopolistic firms on the prices, level of output and the social welfare in the economy. Standard text book models of monopolistic and oligopolistic firms (e.g., Mas-Collel, Winston and Green, 1995) predict lower output and higher prices as compared to the case of competitive, price-taking firms. However, since the seminal paper of Coase (1972) it has been known that the durability of production good renders the profit maximization objective of a monopolist time-inconsistent, destroying monopolistic profit as the decision-making interval shrinks to zero when the pre-commitment to low production is impossible. The reason is that in the absence of pre-commitment a durable goods monopoly today competes with itself tomorrow, which pushes the prices down.

The importance of Coase’s observation for finance comes through the fact that stocks bear resemblance to durable goods. As a result, a value maximizing monopolistic firm may lose its monopolistic power in setting high stock prices. For a monopolistic firm this point was explored by Kihlstrom (1998) in a two-period framework and Basak and Pavlova (2004) in a multi-period framework. As shown in Basak and Pavlova, the durability of the firm’s stock decreases the valuation of the firm beyond the competitive level and causes the overproduction of the consumption good. Moreover, in line with Coase’s conjecture, the time-inconsistency destroys monopolistic profits as the decision-making interval shrinks to zero.

This paper adopts the idea of Basak and Pavlova to study the effect of multi-period oligopolistic competition and capital accumulation on the valuation and investment decisions of firms. While in Basak and Pavlova the firm makes static decisions on labor and production at each period, in this paper we explicitly model investment decisions and capital accumulation. As in their paper, the firms have an economy wide impact which is modeled by assuming one consumption good in the economy. The firms produce non-tradable capital good from which the consumption good is produced via a decreasing returns to scale technology. At each point of time the firms make decisions concerning the level of capital good investment and the level of consumption good output. The firms are stock price maximizers and realize their impact on the state prices by affecting the marginal utility of a representative consumer with CRRA utility function who is the only shareholder of all the firms in the economy. We investigate both the case when the firms can pre-commit to their production plan and when they use time-consistent strategies. The pre-commitment case is analyzed in a continuous time framework with stochastic technologies while time-consistent case is solved in discrete time without uncertainty.

The contribution of this paper is twofold. First, we study the problem of time-inconsistency of the value maximization for the case of dynamic oligopolistic competition while previous literature has only focused on monopolistic producers. Second, we model investment decisions and characterize the effect of time-inconsistency, industry competition and the length of the decision-
making interval on the investment policy of the firm. The investment policy of the firm has long been an important issue in financial and economic literature. In the paper we explain and quantify the effect of the inability to pre-commit on the investment and production decisions of the firm. We show, that the inability to pre-commit leads to the disinvestment of capital and completely destroys the value of the firm in a continuous time limit. However, the industry competition helps mitigate the effect of time-inconsistency. We quantify how intense should be the competition in order to eliminate the adverse effects of time-inconsistency. Our results imply that for a small decision-making interval the accumulation of capital is possible only in the case of high industry competition, low risk aversion or high returns to capital investments. We also derive conditions that characterize the investment policy in the presence of time-inconsistency.

In line with Basak and Pavlova (2004), the firms find it optimal to depress current prices of output in order to boost today’s stock price. The reason is that being unable to pre-commit the firm competes not only with the other firms but also with itself as of next period. Since the firm in the future makes a choice which is sub-optimal from today’s perspective, its valuation will be lower than in the case of pre-commitment. As a result, the firm boosts current production beyond competitive level decreasing today’s marginal utility of a consumer which leads to higher valuation of its future profits. This overproduction is in sharp contrast with the pre-commitment solution which coincides with the competitive one when the relative risk aversion of the representative consumer is low.

The time-inconsistency effect that we study in this paper is similar to that in the theory of hyperbolic discounting (e.g., Laibson, 1997 and Harris and Laibson, 2001). As in the case of hyperbolic discounting, in our setting the long-term discount rate between two distant periods $t$ and $t+1$ will be different form the short-term period discount rate as of time $t$. This happens because the firm finds it optimal to revise its production policy at later dates, which changes state prices between any two dates in the future. Related papers also are Basak (1997) and DeMarzo and Urosevic (2006) which study the effect of large, non price-taking traders on asset pricing.

The paper is also related to a number of works that study the oligopolistic competition in a dynamic setting. Maskin and Tirole (1988) develop a theory of dynamic oligopolistic competition of firms that maximize the sum of discounted profits. They study the Markov Perfect Equilibrium (MPE) of an alternating move, infinite horizon duopoly game and derive the system of Bellman equations characterizing the time-consistent MPE. In a companion paper Maskin and Tirole (1987) obtain a closed form solution to an alternating move duopoly model for the case of quadratic payoff functions, showing that the output in the dynamic case is higher than in the static one. However, these two papers do not model the investment decisions of the firms and their ability to affect state prices in the economy.

Reynolds (1991) studies a linear-quadratic differential game of oligopolistic competition of firms with strategic capacity investment and convex adjustment cost. However, the paper does not model the consumer behavior taking linear demand function as given. Thus, today’s produc-
tion does not affect future state prices, as is the case in our model, and the objective function in Reynolds’s model is time-consistent. Dockner and Sørger (1996) and Dockner and Nishimura (2005) study the dynamic capital accumulation games between two agents. Even though these papers show the implications of competition for the investment decisions the setup is different from ours in that the agents maximize expected utility rather than profits and share the single stock of capital.

The paper is organized as follows. In Section 2 we describe general economic setup. In Section 3 we derive the competitive partial and general equilibria in continuous time with stochastic production technologies. In Section 4 we analyze the pre-commitment case, also in continuous time. Finally, Section 5 studies discrete time, no uncertainty time-consistent problem for oligopolistic firm. The conclusion follows.

2. The Economy

We consider an economy with a representative agent and $n$ firms. The agent is an expected utility maximizer and the only shareholder of all the firms in the economy. She maximizes her lifetime utility function subject to intertemporal budget constraint. Throughout the paper we assume that the agent has CRRA utility function with parameter $\gamma > 0$ for constant relative risk aversion. Assuming that the consumer faces a complete market her problem can be represented as follows (Cox and Huang, 1989; Karatzas, Lehoczky and Shreve, 1990):

$$\max E_0 \left[ \int_0^T e^{-\lambda t} u(c_t) dt \right],$$  \hspace{1cm} (1)

subject to $$E_0 \left[ \int_0^T \xi_t c_t dt \right] \leq E_0 \left[ \int_0^T \xi_t y_t dt \right],$$  \hspace{1cm} (2)

where

$$u(c_t) = \begin{cases} 
    c_t^{1-\gamma} / (1-\gamma) & \text{for } \gamma > 0 \text{ and } \gamma \neq 1, \\
    \ln(c_t) & \text{for } \gamma = 1,
\end{cases}$$

$y_t$ denotes the aggregate dividend paid out by firms to the shareholder and $\xi_t$ stands for the state price.

We consider a Lucas (1978) type economy where in equilibrium aggregate dividend paid out at time $t$ equals consumption of the representative agent, $c_t = y_t$. However, unlike the Lucas’s economy, the aggregate production is endogenous rather than exogenous. Each firm is endowed with a technology that produces consumption good from an intermediate capital good. Moreover, the firm can produce more intermediate good from intermediate good using a stochastic technology. At each point of time the firms make decision on the amount of intermediate good to be reinvested and the amount of consumption good to be produced and paid out as a divided to the shareholder.
The consumption good is produced from capital good using the following decreasing returns to scale technology:

$$g_t = \varepsilon_t \pi_t^{1-\delta},$$

(3)

where $0 < \delta < 1$, $\pi$ denotes the amount of capital good used in the production and $\varepsilon_t$ is a multiplicative shock, the same for all the firms, that follows a geometric Brownian motion:

$$d\varepsilon_t = \mu \varepsilon_t dt + \sigma \varepsilon_t dw_{1t}.$$  

(4)

The capital of an individual firm evolves according to the following stochastic differential equation:

$$dk_t = (\alpha k_t - \pi_t) dt + \sigma k_t dw_{2t},$$

(5)

where $\alpha > 0$. This technology is a straightforward stochastic generalization of a conventional constant returns to scale technology. The variance of the process is proportional to the aggregate level of capital which means that if one unit of capital produces $(1 + \alpha dt) + \sigma k_t dw_{1t}$ units of capital as $dt$ elapses, $k_t$ units will produce $(1 + \alpha dt)k_t + \sigma k_t dw_{1t}$ units, which gives rise to process (5).

The aggregate production, $y$, follows the process

$$dy_t = \mu y_t dt + \sigma y_t dw_{3t},$$

(6)

where $\mu y_t$ and $\sigma y_t$ are determined in equilibrium. In what follows by $\rho_{ij}$ we denote the correlations between Brownian motions $w_i$ and $w_j$.

The objective of the firm is to maximize its market value. To justify the value maximization we note that the indirect utility function of the representative agent should be an increasing function of her total wealth given by the expression on the right hand side of (2). As a result, if the representative shareholder in this economy acts as a price taker she may find the value maximizing policy of the firms to be in congruence with her own interests. This motivates the firm to maximize its value given by the following formula which holds in a Lucas type economic setting:

$$J(k_t, \varepsilon_t, t) = E_t \left[ \int_t^T e^{-\lambda(s-t)} \frac{u'(y_s)}{u'(y_t)} \varepsilon_s^s \pi_s^{1-\delta} ds \right].$$

(7)

The firm maximizes (7) subject to (5). In a competitive economy the producers take the process for $y_t$ as exogenous and maximize their objective function ignoring their own impact on the aggregate production. Since all the producers are identical, in a competitive equilibrium we require that processes $ng_t$ and $y_t$ coincide. In the case of imperfect competition the firms additionally take into account their own impact on the aggregate level of production, $y$. 

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3. Competitive Equilibrium

In this Section we solve explicitly for the unique competitive equilibrium in the economy. By a competitive equilibrium we mean an equilibrium in which all producers act as price takers, ignoring their impact on the aggregate production. Since all the firms in the economy are identical we concentrate only on symmetric equilibria.

Definition. A competitive symmetric equilibrium in an economy with \( n \) competitive identical firms and one representative consumer is a set of strategies \( \{ \pi_t^*, t \in [0, T] \} \), the same for all the firms in the economy, which maximizes the objective function of the firm (7) for each \( t \) subject to (5), taking \( y_t = n \varepsilon_t (\pi_t^*)^{1-\delta} \) as exogenous.1

To derive the competitive equilibrium we first solve a partial equilibrium model in which the process for aggregate output is exogenous and derive the process for individual firm’s output in the form:

\[
dg_t = \mu_g(t; \mu_{yt}, \sigma_{yt}) g_t dt + \sigma_g(t; \mu_{yt}, \sigma_{yt}) g_t dw_{4t}.
\]

(8)

We conjecture that functions \( \mu_{yt} \) and \( \sigma_{yt} \) depend only on time. Later on we will show that this is indeed the case in equilibrium. Once we explicitly derive functions \( \mu_g(t; \mu_{yt}, \sigma_{yt}) \) and \( \sigma_g(t; \mu_{yt}, \sigma_{yt}) \) equilibrium condition \( g_t = y_t \) allows to pin down \( \mu_{yt} \) and \( \sigma_{yt} \) as a fixed point of the following mapping:

\[
\mu_g(t; \mu_{yt}, \sigma_{yt}) = \mu_{yt},
\]

(9)

\[
\sigma_g(t; \mu_{yt}, \sigma_{yt}) = \sigma_{yt}.
\]

(10)

Moreover, we show that a Brownian motion \( w_{4t} \) form (8) is a linear combination of \( w_{1t} \) and \( w_{2t} \). This allows to determine \( w_{3t} \) form (6) since our definition of equilibrium entails that \( w_{3t} = w_{4t} \). The last condition allows to find \( \rho_{13} \) and \( \rho_{23} \) as functions of the parameters of the model.

Thus, finding a competitive equilibrium amounts to finding the process for individual output, \( g_t = \varepsilon_t \pi_t^{1-\delta} \), where \( \pi_t \) maximizes the right-hand side of the following Bellman equation for the value of the firm \( J(k_t, \varepsilon_t, t) \):2

\[
J(k_t, \varepsilon_t, t) = \max_{\varepsilon_t} \left\{ \varepsilon_t \pi_t^{1-\delta} dt + e^{-\lambda dt} E_t \left[ \frac{u'(y_{t+dt})}{u'(y_t)} J(k_{t+dt}, \varepsilon_{t+dt}, t + dt) \right] \right\},
\]

(11)

\[
J(k_T, \varepsilon_T, T) = 0,
\]

(12)

1Further in this Section without loss of generality we set \( n = 1 \), since the number of the firms does not affect the formula for the value of the firm in the case of CRRA utility function.

2The value of the firm \( J(k_t, \varepsilon_t, t) \) is independent of \( y_t \) because due to our assumption that \( \mu_{yt} \) and \( \sigma_{yt} \) depend on time only, the random variable \( (\varepsilon_t/y_t)^{-\gamma} \) is log-normally distributed with the mean and variance independent of \( y_t \).
which can be expressed more conveniently in the following differential form:

$$0 = \max_{\pi_t} \left\{ \varepsilon_t \pi_t^{1-\delta} dt - \lambda J(k_t, \varepsilon_t, t) dt + E_t \left[ \frac{d(u'(y_t) J(k_t, \varepsilon_t, t))}{u'(y_t)} \right] \right\}, \quad (13)$$

$$J(k_T, \varepsilon_T, T) = 0.$$ 

Proposition 1 characterizes the partial equilibrium of the model in which the aggregate output in the economy is exogenous. To this end, we solve Bellman equation (13) subject to (5) where the aggregate output evolves according to (6).

**Proposition 1.** In the partial equilibrium in which the exogenous process for the aggregate output in the economy is given by (6) the value of the stock price maximizing firm is equal to $J(k_t, \varepsilon_t, t) = S(t) \varepsilon_t k_t^{1-\delta}$, where function $S(t)$ is the solution to the following ordinary differential equation:

$$S'(t) + a(t) S(t) + \delta S(t)^{-\frac{1-\delta}{\delta}} = 0, \quad S(T) = 0, \quad (14)$$

where function $a(t)$ is given by

$$a(t) = -\lambda - \gamma \mu_t + \frac{\gamma(1+\gamma) \sigma_{yt}^2}{2} + (1-\delta)(\alpha - \gamma \sigma_k \sigma_{yt} \rho_{23}) +$$

$$+ \mu \varepsilon - \gamma \sigma_{yt} \sigma_{\varepsilon} \rho_{13} - \delta(1-\delta) \frac{\sigma_{\varepsilon}^2}{2} + (1-\delta) \sigma_{\varepsilon} \sigma_k \rho_{12}.$$

The output of the value maximizing firm is given by

$$g_t = S(t)^{-\frac{1-\delta}{\delta}} \varepsilon_t k_t^{1-\delta}. \quad (15)$$

**Proof:** See Appendix.

Proposition 1 gives the expression for the value of the firm in the partial equilibrium. Next step is to endogenize the process for aggregate production following the argument outlined above.

**Proposition 2.** In the competitive equilibrium the value of the stock price maximizing firm is equal to $J(k_t, \varepsilon_t, t) = S(t) \varepsilon_t k_t^{1-\delta}$, where function $S(t)$ satisfies ordinary differential equation

$$S'(t) + AS(t) + \delta S(t)^{1-\frac{1}{\delta}} = 0, \quad S(T) = 0, \quad (16)$$

where constant $A$ is given by the following expression:

$$A = \frac{-\delta}{\delta + \gamma(1-\delta)} \left( a_0 + a_{11} \sigma_{k}^2 - 2a_{12} \sigma_k \sigma_{\varepsilon} + a_{22} \sigma_{\varepsilon}^2 \right), \quad (17)$$

where $a_0 = \lambda - (1-\gamma)(1-\delta) \alpha - (1-\gamma) \mu \varepsilon$, $a_{11} = (1-\delta)(1-\gamma)(\gamma(1-\delta) + \delta)/2$, $a_{22} = (1-\gamma) \gamma/2$, $a_{12} = (1-\delta)(1-\gamma)^2 \rho_{12}/2.$
Moreover, the unique solution to the differential equation (16) is given by:

$$S(t) = \begin{cases} \left( e^{\frac{A}{2}(T-t)} - 1 \right) \frac{A}{\pi}, & \text{if } A \neq 0, \\ (T-t)^{\delta}, & \text{if } A = 0. \end{cases}$$ (18)

In a competitive equilibrium the aggregate production process is given by (6) where

$$\mu_y(t) = (1 - \delta)(A/\delta + \alpha - \delta \sigma_k^2/2 + \sigma_{\delta} \sigma_{\rho_{12}}) + \mu_\varepsilon,$$

$$\sigma_y(t) = \sqrt{\sigma_{\delta}^2 + (1 - \delta)^2 \sigma_k^2 + 2(1 - \delta) \sigma_k \sigma_{\rho_{12}},}$$

$$dw_{3t} = (\sigma_{\delta} dw_{1t} + (1 - \delta) \sigma_k dw_{2t})/\sigma_y.$$

**Proof:** See Appendix.

**Corollary 1 (Infinite Horizon).** If $A < 0$ there exists unique stationary equilibrium in which the value of the firm is given by $J(k_t, \varepsilon_t, t) = S \varepsilon_t k_t^{1-\delta}$, where the constant $S$ is given by the following expression

$$S = \frac{\delta + \gamma (1 - \delta)}{a_0 + a_1 \sigma_k^2 - 2 a_2 \sigma_k \sigma_{\varepsilon} + a_3 \sigma_{\varepsilon}^2}.$$

**Proof:** The proof easily follows from Proposition 2.

The competitive equilibrium derived in this Section is used as a benchmark to compare with the pre-commitment oligopolistic equilibrium of the next Section. However, the model developed here might prove useful in analyzing the effect of uncertainty on the value of the firm. The expression for the value of the firm in the stationary equilibrium shows some nontrivial interaction between production shocks. In this sense our model is close to the models in Abel (1983) and Abel (1985) which explore the effect of output and factor price uncertainties on the production. However, the analysis in these papers assumes that the asset payoffs and discount factor are independent. Craine (1989) in a discrete time model explores the effect of output price uncertainty on the firm’s demand for capital explicitly accounting for the interdependency of the discount factor and the asset payoffs. Our model of a competitive equilibrium endogenizes the discount factor’s effect on the asset valuation in continuous time in a different economic setting. The distinction of our model from the models of Abel and Craine is that the sources of uncertainty are two stochastic production technologies. The first technology produces the capital good which is used to produce the consumption good via the second technology. As a result, negative shock to capital in our model can be partially offset by a positive shock to the production of consumption good, which renders the effect of uncertainty on the value of the firm ambiguous. Moreover, we obtain a closed-form solution for a representative consumer with CRRA utility function whereas Craine provides closed-form solution only for the agent with logarithmic utility over consumption. However, the investigation of the production uncertainty on the output of the firm is beyond the scope of this paper.
4. Oligopolistic Equilibrium: Pre-commitment Case

In this Section we consider oligopolistic competition among \( n \) firms which maximize their value at time \( t = 0 \). The firms make all the production decisions at \( t = 0 \) and are able to pre-commit to their strategy. The model demonstrates that the value maximizing objective of the firms leads to the competitive outcome even in the case of imperfect competition under pre-commitment. Stokey (1981) verifying Coase’s conjecture establishes similar result but for the durable goods monopoly. Bond and Samuelson (1984) show that durable goods monopolist can do better when the pre-commitment is possible. Our model demonstrates the competitive outcome in the case of pre-commitment and with non-durable good.

We assume further that there is no production at \( t = 0 \) and as a state price we take the Lagrange multiplier for the consumer’s budget constraint (2). Then, in equilibrium, the state prices can be computed as

\[
\xi_t = e^{-\lambda t} u'(y_t)/l,
\]

where \( \xi_t \) denotes the state price, and \( l \) denotes the Lagrange multiplier for the consumer’s intertemporal budget constraint. Unlike the case of competitive equilibrium, in the case of oligopolistic competition each producer in the economy realizes impact on the state prices. As a result, firm \( i \) at \( t = 0 \) solves the following problem:

\[
\max_{\pi_i} \mathbb{E}_0 \left[ \int_0^T e^{-\lambda t} (\epsilon_{i,t} \pi_{i,t}^{1-\delta} + y_{-i,t})^{1-\delta} dt \right],
\]

\[
\text{s.t. } dk_{i,t} = (\alpha k_{i,t} - \pi_{i,t}) dt + \sigma_k k_{i,t} dw_{2,t}, \quad k_{i,0} = k_0,
\]

where \( y_{-i,t} \) denotes the aggregate production of all the other firms in the economy. Since the firms in our economy are identical we focus on finding a symmetric equilibrium in this economy.

**Definition.** Symmetric equilibrium in the economy is such a strategy \( \pi_{i,t}^* = \pi_t^* \), the same for all firms, that maximizes objective function of each firm (19) with \( y_{-i,t} = (n-1)\epsilon_t(\pi_t^*)^{1-\delta} \).

**Proposition 3.** If \( \gamma < n \), there exists unique symmetric oligopolistic equilibrium in the economy and the value of the firm is equal to \( J(k_t, \epsilon_t, t) = S(t)\epsilon_t k_t^{1-\delta} \), where \( S(t) \) is given by (18).

**Proof:** See Appendix.

Proposition 3 reveals that the value of the firm in the oligopolistic equilibrium is exactly the same as in the case of competitive equilibrium of the previous Section. The standard prediction in the case of oligopolistic competition is that the production is lower than in the competitive outcome (e.g, Mas-Colell, Winston and Green, 1995). Decreasing output has two effects. On one hand, it increases the price of consumption good which has a positive effect on the firms’ profits.
On the other hand, lower quantity sold has a negative effect on profits. In our model these two effects offset each other leading to a competitive socially optimal outcome.

However, the symmetric equilibrium exists only for $\gamma < n$. To understand this better consider the case of a monopolistic firm. From (19) the objective function of the firm is to maximize

\[
E_0 \left[ \int_0^T e^{-\lambda t} (\varepsilon_1 \pi_1^{1-\delta})^{1-\gamma} dt \right]
\]

subject to dynamic constraint (20). Observe that this problem coincides with the social planner’s problem if $\gamma < 1$. When $\gamma > 1$ the monopoly’s problem is tantamount to minimizing the utility of the representative agent subject to (20). To see this, notice that the maximization problem of the firm can be rewritten as

\[
\max \left\{ (1 - \gamma) E_0 \left[ \int_0^T e^{-\lambda t} \left( \varepsilon_1 \pi_1^{1-\delta} \right)^{1-\gamma} dt \right] \right\}.
\]

Since $1 - \gamma$ is negative when $\gamma > 1$, the monopolist will be minimizing the utility function of the representative agent. Thus, oligopolistic firms get extreme monopolistic power in the case when the agents are very risk averse. Even though this result looks unnatural this is due to our oversimplified model. More realistic model should take into account that consumers may have some endowment or have an option to consume close substitute for the good produced by the monopoly. These outside options decrease the oligopolistic power of the firms.

5. Oligopolistic Equilibrium: Time-Consistent Case

In this Section we explore the case when the pre-commitment is impossible. The firm solves for the time-consistent strategy by backwards induction starting at $t = T$. Making decisions at an intermediate date and being unable to pre-commit the firm treats its future strategies as given. As it is shown in Basak and Pavlova (2004) in a different discrete time setting, when the firm is unable to pre-commit its value converges to zero as the decision-making interval shrinks to zero. In this Section we show that similar result obtains in our model. Since we want to trace the effect of decision-making interval on the firms’ valuation, we consider a discrete time model. Moreover, the model of this Section is deterministic, which allows to quantify the effect of competition and time-inconsistency on the investment policy of the firm.

Firm’s production at time $t$ affects both the amount of capital available next period and the consumer’s valuation of future dividends. Increasing time-$t$ production the firm reduces future capital but increases the valuation of future payoffs by making the consumer more willing to spread consumption which increases her valuation of future dividends. The capital and consumption good productions are:

\[
k_t = (1 + \alpha h) k_{t-1} - \pi_{t-1} h,
\]

\[
g_t = \pi_{t-1}^{1-\delta},
\]

(21)
where $h$ denotes the decision-making interval and equation (21) is a deterministic version of equation (5). Then, we get the Bellman equation for each oligopolistic competitor in the economy:

$$J_i(k_t, t; h) = \max_{\pi_{it}} \left\{ \pi_{it}^{1-\delta} h + (1 - \lambda h) \frac{(\pi_{it}^{1-\delta} + y_{-it}(t+1))^{-\gamma}}{(\pi_{it}^{1-\delta} + y_{-it})^{-\gamma}} J_i(k_{t+1}, t+1; h) \right\},$$

(22)

where $y_{-it} = \sum_{j \neq i} \pi_{jt}^{1-\delta}$, $\lambda$ is a parameter reflecting time preferences and $t = 1...N$. Parameters $N$ and $h$ are such that $T = hN$. As in the previous Section, $(\pi_{it}^{1-\delta} + y_{-it})^{-\gamma}/(\pi_{it-1}^{1-\delta} + y_{-it-1})^{-\gamma} = \xi_{it}/\xi_{i,t-1}$. First, for simplicity, we derive results for the case of a monopolistic firm and then point out how the results can be extended to the case of oligopolistic competition. When $n = 1$ the Bellman equation will be rewritten as follows:

$$J(k_t, t; h) = \max_{\pi_{it}} \left\{ \pi_{it}^{1-\delta} h + (1 - \lambda h) \frac{(\pi_{it}^{1-\delta} + y_{-it})^{-\gamma}}{(\pi_{it}^{1-\delta})^{-\gamma}} J(k_{t+1}, t+1; h) \right\},$$

(23)

Equation (23) shows that the time-inconsistent nature of the problem is due to the changing state prices. If the agent at time $t$ pre-commits to maximize the function

$$\pi_{it}^{1-\delta} h + (1 - \lambda h) \frac{(\pi_{it}^{1-\delta})^{-\gamma}}{(\pi_{it}^{1-\delta})^{-\gamma}} \sum_{\tau=t+1}^{T} \frac{(\pi_{i+1}^{1-\delta})^{-\gamma}}{(\pi_{i}^{1-\delta})^{-\gamma}} \pi_{i}^{1-\delta},$$

then, her time-$t$ choice will be sub-optimal sitting at time $t+1$, since the solution to time-$(t+1)$ problem, which is to maximize

$$\sum_{\tau=t+1}^{T} \frac{(\pi_{i+1}^{1-\delta})^{-\gamma}}{(\pi_{i}^{1-\delta})^{-\gamma}} \pi_{i}^{1-\delta},$$

does not lead to the same strategy as the maximization of time-$t$ objective. As a result, at time $t + 1$ the agent will be facing state prices which are different from the state prices that arise in time-$t$ optimization. Our time-inconsistency problem is similar to the case of the utility functions with hyperbolic discounting. As in the hyperbolic discounting case (e.g., Laibson, 1997 and Harris and Laibson, 2001), the discount which the monopolist places on future events changes in the course of time as those future events draw nearer.

The oligopolist’s choice tomorrow will be sub-optimal from the standpoint of today’s objective. As a result, the firm’s value today will be lower than in the case of pre-commitment. This may prompt the monopolist to increase production today, which increases the valuation of the future profits and offsets the loss in the firm’s value due to the sub-optimality of her choice in the future. Since the firm takes its future actions as given time-consistent strategy can be interpreted as a subgame-perfect Nash equilibrium in an intrapersonal game as in Laibson (1997), Harris and Laibson (2001) and Benabou and Tirole (1999). The firm in this game competes with itself as of tomorrow choosing best response to its own future actions. As a result, it overproduces as compared to the pre-commitment case. Naturally, the overproduction gets exacerbated as the time-horizon increases and the decision-making interval shrinks. We show that the value of the
firm converges to zero as the decision-making interval shrinks to zero. The explanation is that because of overproduction, the oligopolist depletes the stock of capital very quickly. This being the case, consumers consume much more in earlier dates than in the future and the inability to smooth consumption across periods decreases their valuation of the firm. Propositions 4 and 5 report the results. Even though Proposition 4 is a corollary of Proposition 5, we state and prove them separately because the proof in monopolistic case is more simple and straightforward, than in the general oligopolistic case.

**Proposition 4.**

1. If \( \gamma < 1 \) there exists unique equilibrium in the economy with a monopolistic firm in which \( \pi_t = C_t k_t \), where \( C_t \) is a function of time only. \( C_t \) can be found from the following recursive relation:

\[
(1 - \lambda h)(1 - \gamma)(1 + \alpha h) C_t^{(1-\gamma)(1-\delta)-1} \left( \frac{1 + \alpha h}{C_t - h} \right)^{\gamma (1-\delta)-1} \left[ \frac{h - \gamma}{C_t - h} \right] = 1, \tag{24}
\]

\( C_T = \frac{1 + \alpha h}{h}. \)

2. Denote \( \Delta = (1 - \gamma)(1 + \alpha h)(1 - \lambda h) \). Then, if \( \Delta < 1 \), \( J(k_t, t; h) \to 0 \) as \( h \to 0 \).

3. If \( \Delta < 1 \) capital \( k_t \) is decreasing in \( t \), if \( \Delta > 1 \) and \( T \) is sufficiently large, \( k_t \) is a hump-shaped function increasing on some interval \([0, T_0]\) and decreasing on \((T_0, T]\), where \( 0 < T_0 < T \).

**Proof:** See Appendix.

**Proposition 5 (Oligopolistic Competition).**

1. If \( \gamma < n \) there exists unique symmetric equilibrium in the economy with oligopolistic competition in which \( \pi_t = C_t k_t \), where \( C_t \) is a function of time only. \( C_t \) can be found from the following recursive relation:

\[
(1 - \lambda h)(1 - \gamma)(1 + \alpha h) C_t^{(1-\gamma)(1-\delta)-1} \left( \frac{1 + \alpha h}{n C_t - h} \right)^{\gamma (1-\delta)-1} \left[ \frac{h - \gamma}{n C_t - h} \right] = 1, \tag{25}
\]

\( C_T = \frac{1 + \alpha h}{h}. \)

2. Denote \( \Delta_n = (1 - \gamma)(1 + \alpha h)(1 - \lambda h) \). If \( \Delta_n < 1 \), \( J(k_t, t; h) \to 0 \) as \( h \to 0 \).

3. If \( \Delta_n < 1 \) capital \( k_t \) is decreasing in \( t \), if \( \Delta_n > 1 \) and \( T \) is sufficiently large, \( k_t \) is a hump-shaped function increasing on some interval \([0, T_0]\) and decreasing on \((T_0, T]\), where \( 0 < T_0 < T \).

**Proof:** See Appendix.

Propositions 4 and 5 show that the results in the time-consistent case are in striking difference with the pre-commitment case. While in the pre-commitment case the solution of the oligopolist’s
problem is well defined, which we can prove in a very general framework, without pre-commitment
the overproduction makes the firm less attractive to the consumer, which drives the value of the
firm to zero in a continuous time limit.

To understand the intuition behind $\Delta_n$ we rewrite the inequality $\Delta_n > 1$ for small values of
$h$ as follows:

$$(\alpha - \lambda)h > \frac{1}{1 - \gamma/n} - 1. \tag{26}$$

The inequality (26) in a concise way summarizes the tradeoff between capital accumulation and
time effects. Naturally, the firm accumulates capital if $\alpha$ is sufficiently high. High $\alpha$ implies that
production technology is efficient and the depletion of capital is costly. Note that even in the
competitive case when $n$ is very high, the accumulation of capital occurs only if $\alpha > \lambda$ since
otherwise the representative consumer will be too impatient to consume and will be better off
gradually depleting the stock of capital. The inequality (26) is violated for small $h$ in which case
the firm chooses not to accumulate capital. Besides that, the low level of risk aversion relaxes
the problem of time-inconsistency, since the consumer gets less willing to smooth consumption,
and as a result, less responsive to the production policy of the firms.

Another result prompted by the inequality (26) is that it is satisfied when the industry
competition is very high. If $n$ is sufficiently high the right hand side of the inequality will be
close to 0, and the inequality will be satisfied. This result suggests that the industry competition
mitigates adverse effects of time-inconsistency. When the competition is high the firms lose their
impact on the state prices and their inability to manipulate with the marginal utility of the
consumer eliminates the time-inconsistency effect. Besides general intuition our inequality gives
some qualitative results. It shows that for the accumulation of capital to take place the number
of competitors in the economy should exceed some critical level $n^*$, proportional to $1/h$ for small
$h$. This helps understand how intense should be the competition in order to eliminate the effect
of time-inconsistency. The discussion above allows to conclude that $\Delta_n$ summarizes the impact
of time, competition and productivity on the accumulation of capital.
6. Conclusion

In this paper we consider a production economy with consumption and capital goods, a representative consumer and value maximizing oligopolistic firms. The firms in this economy recognize their impact on state prices and manipulate marginal utilities of the consumer in order to increase the valuation of their stocks, which renders their objective function time-inconsistent. Besides competing with other firms each firm competes with itself as of next period which destroys its value due to the overproduction of consumption good as the decision-making interval shrinks to zero. We have also shown how investment decisions of the firms are affected by time-inconsistency and industry competition, establishing the conditions under which oligopolistic firms choose to invest in capital good. Our results imply that with a small decision-making interval capital accumulation will be observed only in the case of high industry competition.

We have also analyzed the competitive and pre-commitment equilibria in the economy. The analysis of these two cases was carried out in a rather general case of continuous time decision-making with stochastic production technologies. We have established that under low enough level of risk aversion the value maximizing policy of the firm leads to the competitive outcome. In the paper we tried to keep the setting as simple as possible. More realistic model should include at least two consumption goods and production sectors in the economy, since one of them being competitive may limit the extent to which the oligopolists manipulate with marginal utilities of consumers. Another crucial extension would be to account for the labor market.

Future research in the direction outlined in this paper may focus on commitment devices which
help resolve time-inconsistency problem. One of the possible solutions might be the increase of debt in the capital structure of the firm. The necessity to repay debt may help the firm to pre-commit against overproduction, since otherwise, overproduction will cause the depletion of the capital and inability to service debt obligations. As a result, the possibility of future default will be priced, decreasing the current value of the firm.
Proof of Proposition 1.

The prove of Proposition 1 amounts to explicitly solving the Bellman equation (13). In order to obtain HJB differential equation we first apply Ito’s lemma to \( d(u'(y_t)J(k_t, \varepsilon_t, t)) \). The Ito’s lemma applied to \( du'(y_t) \) and \( dJ(k_t, \varepsilon_t, t) \) gives the following expressions:

\[

du'(y_t) = (u''(y_t)\mu_y y_t + u'''(y_t)\sigma_y^2 y_t^2/2) \, dt + u''(y_t)\sigma_y y_t \, dw_t,
\]

\[
dJ(k_t, \varepsilon_t, t) = (J_t + J_k(\alpha k_t - \pi_t) + J_\varepsilon \varepsilon_t \mu_\varepsilon + J_{kk} \sigma_k^2 k_t^2 + J_{\varepsilon\varepsilon} \sigma_\varepsilon^2 \varepsilon_t^2 + J_{k\varepsilon} \sigma_k \varepsilon_t \sigma_\varepsilon \rho_{12} \varepsilon_t) \, dt + J_k \sigma_k k_t \, dw_t + J_\varepsilon \varepsilon_t \, dw_t,
\]

where by \( J_t, J_k, J_\varepsilon, J_{kk}, J_{\varepsilon\varepsilon} \) and \( J_{k\varepsilon} \) we denote respective partial derivatives of function \( J(k_t, \varepsilon_t, t) \). Using the chain rule

\[
d(u'(y_t)J(k_t, \varepsilon_t, t)) = J(k_t, \varepsilon_t, t)du'(y_t) + u'(y_t)dJ(k_t, \varepsilon_t, t) + dJ(k_t, \varepsilon_t, t)du'(y_t)
\]

and taking into account the properties of CRRA utility function we obtain the following:

\[
E_t \left[ \frac{d(u'(y_t)J(k_t, \varepsilon_t, t))}{u'(y_t)} \right] = J_t + J_k(\alpha k_t - \pi_t - \gamma \sigma_y \sigma_k \rho_{23} k_t) + J_\varepsilon \varepsilon_t \mu_\varepsilon - \gamma \sigma_\varepsilon \sigma_y \rho_{13} \varepsilon_t + J_{kk} \sigma_k^2 k_t^2/2 + J_{\varepsilon\varepsilon} \sigma_\varepsilon^2 \varepsilon_t^2/2 + J_{k\varepsilon} \sigma_k \varepsilon_t \sigma_\varepsilon \rho_{12} + J(-\gamma \mu_y + \frac{\gamma(1+\gamma)\sigma_y^2}{2}).
\]

Substituting this expression back into Bellman equation (13) yields the following:

\[
0 = \max_{\pi_t} \{ \varepsilon_t \pi_t^{-1-\delta} - \pi_t J_k + J_t - \lambda J + J_k(\alpha - \gamma \sigma_y \sigma_k \rho_{23} k_t) + J_\varepsilon \varepsilon_t \mu_\varepsilon - \gamma \sigma_\varepsilon \sigma_y \rho_{13} \varepsilon_t + J_{kk} \sigma_k^2 k_t^2/2 + J_{\varepsilon\varepsilon} \sigma_\varepsilon^2 \varepsilon_t^2/2 + J_{k\varepsilon} \sigma_k \varepsilon_t \sigma_\varepsilon \rho_{12} + J(-\gamma \mu_y + \frac{\gamma(1+\gamma)\sigma_y^2}{2}) \}.
\]

Now we obtain the first order condition with respect to \( \pi_t : (1-\delta)\varepsilon_t \pi_t^{-\delta} = J_k. \) This first order condition allows to express \( \pi_t \) as a function of \( \varepsilon_t \) and \( J_k. \) Then, the substitution of \( \pi_t \) into the Bellman equation will give us the HJB partial differential equation. Though we do not write out explicitly this partial differential equation next we prove that the solution has the form \( J(k, \varepsilon, t) = S(t)\varepsilon k_t^{1-\delta}. \) The first order condition and the functional form of the solution to our Bellman equation imply the following expression for \( \pi_t \):

\[
\pi_t = S(t)^{-\frac{1}{\delta}} k_t.
\]

Substituting (28) and \( J(k, \varepsilon, t) = S(t)\varepsilon k_t^{1-\delta} \) into Bellman equation (27) gives us equation (14). In this way we conclude, that the solution to Bellman equation (27) is indeed given by function \( J(k, \varepsilon, t) = S(t)\varepsilon k_t^{1-\delta} \) where \( S(t) \) satisfies ordinary differential equation (14). The output of the firm will be given by \( g_t = S(t)^{-\frac{1-\delta}{\delta}} \varepsilon_t k_t^{1-\delta} \), since by definition \( g_t = \varepsilon_t \pi_t^{-1-\delta}. \)

Q.E.D.
Proof of Proposition 2.

Proposition 1 gives explicit solution to the partial equilibrium model. In the competitive case each firm takes the output of all the other firms in the economy as given. As discussed in Section 3, we conjecture that \( \mu_{yt} \) and \( \sigma_{yt} \) are the functions of time only which we denote as \( \mu_y(t) \) and \( \sigma_y(t) \) respectively. This prompts the following strategy for determining the value of the company in equilibrium. First, we derive the process for the output of the firm in the following form:

\[
dg_t = \mu_y(t; \mu_y(t), \sigma_y(t))g_tdt + \sigma_y(t; \mu_y(t), \sigma_y(t))g_tdw_{4t}. (29)
\]

After that, using the fact that in equilibrium \( g_t = y_t \) we determine endogenous quantities \( \mu_y(t), \sigma_y(t), \rho_{13} \) and \( \rho_{23} \) from the following conditions:

\[
\begin{align*}
\mu_y(t; \mu_y(t), \sigma_y(t)) &= \mu_y(t), \\
\sigma_y(t; \mu_y(t), \sigma_y(t)) &= \sigma_y(t), \\
dw_{4t} &= dw_{3t}.
\end{align*}
\]

Given the production decisions of all the other firms in the economy the output of the firm can be found using formula (15). According to this formula, the output of the company is \( y_t = S(t)^{-\frac{1}{2}} \varepsilon_t k_t^{1-\delta} \). Applying Ito’s lemma to \( g_t \) we get the following geometric Brownian motion process:

\[
dg_t = g_t \left\{ (1-\delta)\left(-\frac{1}{\delta}S'(t)S(t)^{-\frac{1}{2}} + \alpha - S(t)^{-\frac{1}{2}} \right) - \delta\frac{\sigma_k^2}{2} + \sigma_k \rho_{12} \right\} dt + g_t \left\{ \sigma_k dw_{1t} + (1-\delta)\sigma_k dw_{2t} \right\}.
\]

Now comparing this expression with (29) we obtain that

\[
dw_{3t} = dw_{4t} = \frac{\sigma_k dw_{1t} + (1-\delta)\sigma_k dw_{2t}}{\sqrt{\sigma_k^2 + (1-\delta)^2\sigma_k^2 + 2(1-\delta)\sigma_k \sigma_\varepsilon \rho_{12}}}. (32)
\]

Comparing (31) with (29) and taking into account (30) and (32) we obtain the following formulas:

\[
\begin{align*}
\mu_y(t) &= (1-\delta)\left(-\frac{1}{\delta}S'(t)S(t)^{-\frac{1}{2}} + \alpha - S(t)^{-\frac{1}{2}} \right) - \delta\frac{\sigma_k^2}{2} + \sigma_k \rho_{12} + \mu_\varepsilon, \\
\sigma_y(t) &= \sqrt{\sigma_k^2 + (1-\delta)^2\sigma_k^2 + 2(1-\delta)\sigma_k \sigma_\varepsilon \rho_{12}}.
\end{align*}
\]

Besides that, formula (32) allows us to find \( \rho_{13} \) and \( \rho_{23} \) which we assumed to be exogenous in a partial equilibrium model. Now we derive these quantities in equilibrium:

\[
\begin{align*}
\rho_{13} dt &= dw_{1t} dw_{3t} = \frac{\sigma_\varepsilon + (1-\delta)\sigma_k \rho_{12}}{\sqrt{\sigma_k^2 + (1-\delta)^2\sigma_k^2 + 2(1-\delta)\sigma_k \sigma_\varepsilon \rho_{12}}} dt, \\
\rho_{23} dt &= dw_{2t} dw_{3t} = \frac{\rho_{12} \sigma_\varepsilon + (1-\delta)\sigma_k}{\sqrt{\sigma_k^2 + (1-\delta)^2\sigma_k^2 + 2(1-\delta)\sigma_k \sigma_\varepsilon \rho_{12}}} dt.
\end{align*}
\]
Now it remains to determine $\mu_y(t)$ in the competitive equilibrium. As it can be seen from (33), $\mu_y(t)$ depends on $S(t)$, which is the solution of the ordinary differential equation (14), and, hence, itself depends on $\mu_y(t)$. In order to find the equation for $S(t)$ we just substitute expressions given in (33) into (14). Rearranging the terms we get equation (16) with coefficient $A$ given by the following formula:

$$A = \frac{\delta}{\delta + \gamma(1-\delta)} \left\{ -\lambda + (1-\gamma)(1-\delta)\alpha + \frac{\gamma(1+\gamma)\sigma^2_y}{2} - \gamma(1-\delta)\sigma_k(\sigma_y \rho_{12} + \sigma_y \rho_{23}) + 
(1-\gamma)\mu \varepsilon - \frac{(1-\gamma)\delta(1-\delta)\sigma^2_k}{2} - \gamma \sigma_y \sigma_\varepsilon \rho_{13} + (1-\delta)\sigma_\varepsilon \sigma_k \rho_{12} \right\},$$

where $\sigma_y = \sqrt{(1-\delta)^2 \sigma^2_k + \sigma^2_\varepsilon + 2(1-\delta)\sigma_k \sigma_\varepsilon \rho_{12}}$, $\rho_{23} = ((1-\delta)\sigma_k + \sigma_\varepsilon \rho_{12})/\sigma_y$ and $\rho_{13} = ((1-\delta)\sigma_k \rho_{12} + \sigma_\varepsilon)/\sigma_y$.

Simplifying the above expression for $A$ we get (17).

In order to solve the equation (16) we observe that if $S(t) = H(t)^\delta$ then $H(t)$ solves the Cauchy problem for the following ordinary differential equation:

$$H'(t) + \frac{A}{\delta} H(t) + 1 = 0, \quad H(T) = 0.$$

It can be easily verified, that the unique solution to this problem is $H(t) = (e^{\frac{A}{\delta}(T-t)} - 1)/A$ if $A \neq 0$, and $H(t) = (T-t)$ if $A = 0$. This gives rise to formula (18).

Now in order to complete the characterization of the competitive equilibrium we need to find function $\mu_y(t)$ in equilibrium. To this end, we substitute explicit solution to equation (16) into formula (33) and obtain explicit representation for $\mu_y(t)$:

$$\mu_y(t) = (1-\delta)(A/\delta + \alpha - \delta \sigma^2_k/2 + \sigma_\varepsilon \sigma_k \rho_{12}) + \mu \varepsilon.$$

As conjectured above, $\mu_y(t)$ and $\sigma_y(t)$ appear to be the functions of time only. As in Proposition 1 the output of the firm is equal to $g_t = S(t)^{-\frac{1-\delta}{1-\varepsilon}} \varepsilon k_t^{1-\delta}$. This completes our description of the competitive equilibrium. \(Q.E.D.\)

**Lemma 1** Consider the following stochastic differential equation:

$$dk_t = (\alpha k_t - \pi_t)dt + \sigma_k k_t dw_{2t}.$$ 

Then, the solution to this SDE is given by the following formula:

$$k_t = k_0 e^{(\alpha - \frac{1}{2} \sigma^2) t + \sigma_k w_{2t}} - \int_0^t e^{(\alpha - \frac{1}{2} \sigma^2)(t-\tau) + \sigma_k (w_{2t} - w_\tau)} \pi_\tau d\tau.$$ 

**Proof:**
We look for the solution of the form \( k_t = A(t)x_t \), where \( x_t \) follows the process \( dx_t = \alpha x_t dt + \sigma_k x_t dw_{2t} \). Applying chain rule we get

\[
dk_t = A(t)dx_t + A'(t)x_t dt = A(t)\alpha x_t dt + A(t)\sigma_k x_t dw_{2t} + A'(t)x_t dt = (\alpha k_t + A'(t)x_t) dt + \sigma_k k_t dw_{2t}.
\]

Hence, function \( A(t) \) can be found from the condition: \( A'(t) = -\pi_1/x_t \). Since the process \( x_t \) evolves according to geometric Brownian motion, this process is non-negative and almost everywhere continuous. As a result, we get the solution of the following form:

\[
k_t = (k_0 - \int_0^t \frac{\pi_1}{x_t} dt)x_t.
\]

Since \( x_t \) follows a geometric Brownian motion, \( x_t = x_0 e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma_k w_{2t}} \). Substituting this expression into the formula above we obtain the solution to the problem. \( Q.E.D. \)

**Proof of Proposition 3.**

Since it is rather difficult to solve (19) by means of dynamic programming, we expand the constraint (20) and then solve the static problem. As it follows from Lemma 1,

\[
k_{it} = k_0 e^{(\alpha - \frac{1}{2}\sigma^2)T + \sigma_k w_T} - \int_0^T e^{(\alpha - \frac{1}{2}\sigma^2)(T-\tau) + \sigma_k (w_{2T} - w_{2\tau})} \pi_{1it} d\tau.
\]

Since the capital is always be positive \( E_0 k_{it} \geq 0 \) we get the following budget constraint for the firm:

\[
E_0 \left[ \int_0^T \hat{\xi}_\tau \pi_{it} d\tau \right] \leq k_0,
\]

where \( \hat{\xi}_\tau = e^{(\alpha - \frac{1}{2}\sigma^2)\tau + \sigma_k w_{2\tau}} \). Note also, that given our specification of the production technology \( \pi_{it} \) can never be negative. Since the process for capital is driven by one Brownian motion \( w_2 \), budget constraint (34) is equivalent to the budget constraint in the differential form (20) (Karatzas and Shreve, 1998). As a result, the problem of the firm can be rewritten as follows:

\[
\max_{\pi_{it}} E_0 \left[ \int_0^T e^{-\lambda t(\varepsilon_i \pi_{1it}^{1-\delta} + y_{-it}) - \gamma \varepsilon_i \pi_{1it}^{1-\delta} dt} \right],
\]

s.t. \( E_0 \left[ \int_0^T \hat{\xi}_\tau \pi_{it} d\tau \right] \leq k_0. \)

Now we set up the Lagrangian and derive first and second order conditions for the problem. The Lagrange multiplier is denoted as \( \hat{\ell} \). The first order condition looks as follows:

\[
(1 - \delta)\varepsilon_i \pi_{1it}^{1-\delta} (\varepsilon_i \pi_{1it}^{1-\delta} + y_{-it}) - \gamma - 1 \left( (1 - \gamma)\varepsilon_i \pi_{1it}^{1-\delta} + y_{-it} \right) = \hat{\ell} \hat{\xi}_t e^{\lambda t}.
\]

Since the left-hand side of (37) is strictly positive the Lagrange multiplier is also positive. Thus, the budget constraint of the firm is always binding. In a symmetric equilibrium \( \pi_{it} = \pi_t \) and \( \hat{\ell} \hat{\xi}_t e^{\lambda t} = n \hat{\xi}_t e^{\lambda t} \). Substituting this into first order condition we obtain:

\[
(1 - \delta) n^{-\gamma} (1 - \frac{\gamma}{n}) \varepsilon_i^{1-\gamma} \pi_t^{(1-\delta)(1-\gamma)-1} = \hat{\ell} \hat{\xi}_t e^{\lambda t}.
\]
Since the Lagrange multiplier should be positive we conclude that the solution exists only if inequality $\gamma < n$ is satisfied. Let us denote $t = \hat{t} / [(1 - \delta)n^{\gamma}(1 - \frac{2}{n})]$ and $1 - \beta = (1 - \delta)(1 - \gamma)$.

Then, from (38) we obtain: $\pi_t = (\ell \xi_t e^{\lambda t})^{-1/3}$ where constant $\ell$ can be found from the firm’s budget constraint: $E_0 \int_0^T \xi_t (\ell \xi_t e^{\lambda t})^{-1/3} d\tau = \kappa_0$ . Then, in equilibrium, the value of the firm will be given by the following expression:

$$ V(t) = E_t \left[ \int_t^T e^{-\lambda(t-\tau)} \frac{\varepsilon_t \pi_{1-\delta}(y_{1-\delta} - n)}{\varepsilon_t \pi_{1-\delta}(y_{1-\delta} + y_{-\delta})} d\tau \right] = E_t \left[ \int_t^T e^{-\lambda(t-\tau)} \frac{\varepsilon_t \pi_{1-\delta}(y_{1-\delta} + y_{-\delta})}{\varepsilon_t \pi_{1-\delta}(y_{1-\delta} + y_{-\delta})} d\tau \right] =$$

$$= \varepsilon_t \pi_{1-\delta} E_t \left[ \int_t^T e^{-\lambda(t-\tau)} \left( \frac{\varepsilon_t}{\varepsilon_t} \right)^{\gamma} \left( \frac{\pi_{1-\delta}}{\pi_{1-\delta}} \right)^{(1-\delta)(1-\gamma)} d\tau \right].$$

Thus, the value of the firm does not depend on the number of competing firms. As a result, as $n$ tends to infinity we should obtain competitive case, which is always time-consistent. Hence, the value of the firm is given by $J(k_t, t) = S(t)\varepsilon_t k_{1-\delta}$ where $S(t)$ satisfies (18).

Now we verify the second order conditions. To this end we differentiate left hand side of (37) with respect to $\pi_{it}$ and show that this derivative is negative on the equilibrium path if $\gamma < n$ is satisfied. The second derivative of $\varepsilon_t \pi_{1-\delta} - y_{-\delta} \gamma \varepsilon_t \pi_{1-\delta}^{1-\delta}$ is given by

\begin{align*}
(1 - \delta)^2 (-\gamma - 1) \varepsilon_t \pi_{1-\delta}^{1-\delta} + y_{-\delta}^{1-\delta} - 2 \varepsilon_t \pi_{1-\delta}^{2-\delta} \left( -\gamma - 1 \right) \varepsilon_t \pi_{1-\delta}^{1-\delta} + y_{-\delta}^{1-\delta} + (1 - \gamma) (1 - \delta) \varepsilon_t \pi_{1-\delta}^{2-\delta} \\
(1 - \delta) (\varepsilon_t \pi_{1-\delta}^{1-\delta} + y_{-\delta}^{1-\delta})^{-\gamma - 1} \left( -\delta \varepsilon_t \pi_{1-\delta}^{1-\delta} + (1 - \gamma) \varepsilon_t \pi_{1-\delta}^{1-\delta} + y_{-\delta}^{1-\delta} + (1 - \gamma) (1 - \delta) \varepsilon_t \pi_{1-\delta}^{2-\delta} \right). 
\end{align*}

In equilibrium, $\pi_{it} = \pi_t = (\ell \xi_t e^{\lambda t})^{-1/3}$ and $y_{-\delta} = (n - 1) \varepsilon_t \pi_{1-\delta}^{1-\delta}$ and the firm takes $y_{-\delta}$ as given optimally choosing $\pi_{it} = \pi_t = (\ell \xi_t e^{\lambda t})^{-1/3}$. Now we need to show, that if aggregate production is given by $y_{-\delta} = (n - 1) \varepsilon_t \pi_{1-\delta}^{1-\delta}$ then individual production is maximized for $\pi_{it} = \pi_t = (\ell \xi_t e^{\lambda t})^{-1/3}$. By construction of equilibrium strategy the first order condition is satisfied. Now we verify that the second order condition is also satisfied. To this end, we substitute $\pi_{it} = \pi_t = (\ell \xi_t e^{\lambda t})^{-1/3}$ and $y_{-\delta} = (n - 1) \varepsilon_t \pi_{1-\delta}^{1-\delta}$ into (39) and find conditions on $\gamma$ under which this expression will be negative. Substituting, we obtain the following expression:

$$ (1 - \delta) \varepsilon_t^{-1} \pi_t^{1-\delta} (-\gamma - 1)^{2-\gamma} \left( (1 - \delta) (-\gamma - 1) (1 - \frac{\gamma}{n}) + [-\delta (n - \gamma) + (1 - \gamma) (1 - \delta)] \right). $$

After some simple algebra we get

$$ f(\gamma) = (1 - \delta) (-\gamma - 1) (1 - \frac{\gamma}{n}) + [-\delta (n - \gamma) + (1 - \gamma) (1 - \delta)] =$$

$$= -(1 - \delta) \gamma - (1 - \delta) (-\gamma - 1) \frac{\gamma}{n} - \delta (n - \gamma).$$

It easily follows from the expression above that inequality $\gamma < n$ is sufficient for $f(\gamma)$ to be negative. Hence, we are indeed in equilibrium. Q.E.D.

**Lemma 2** Consider two monotonically increasing continuous functions $f(x)$ and $g(x)$, where $x \in [a, b]$. Suppose the following properties hold:

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1. There exists $x^*$ such that $f(x^*) = g(x^*)$,
2. $g(x) > f(x)$ for all $x > x^*$.
3. For any $x > x^*$ there exists $y$ such that $f(x) = g(y)$.

Consider the following recursion: $f(x_t) = g(x_{t-1})$ which starts at $t = T$ with $x_T = X > x^*$ and is solved backwards. Then, if conditions 1, 2 and 3 and are satisfied, $x_{t-\tau} \to x^*$ as $\tau \to +\infty$.

**Proof:**

First, we show that the sequence $x_t$ is monotonically decreasing. $g(x_{T-1}) = f(x_T) > f(x^*) = g(x^*)$, hence, $x_{T-1} > x^*$. Hence, since property 2 holds, $f(x_T) = g(x_{T-1}) > f(x_{T-1})$ and since function $f(x)$ is monotonically increasing, we obtain that $x_T > x_{T-1}$. Continuing the process we obtain that $x_T > x_{T-1} > ... > x_1 > x^*$. This proves that $x_t$ is a monotonically decreasing function bounded from below, and hence, converges to some value $x^0 \geq x^*$. Moreover, taking limit in $f(x_t) = g(x_{t-1})$ we obtain, that $f(x_0) = g(x_0)$. Now we prove that $x^0 = x^*$. Suppose $x^0 > x^*$. Then, from property 2 we obtain, that $g(x^0) > f(x^0)$, which contradicts the fact that $f(x_0) = g(x_0)$.

**Q.E.D.**

**Proof of Proposition 4.**

In order to prove Proposition 4 we explicitly solve Bellman equation (23) by backwards induction. First, we start from $t = T$. It is easy to see, that at $t = T$ $\pi_T = (1 + \alpha h) k_T$, since the monopolist will spend all the capital to produce consumption good at time $t = T$. As a result, the value of the firm will be $J(k_T, T; h) = \pi_T^{1-\delta} = B_T k_T^{1-\delta}$, where $B_T = (1 + \alpha h)^{1-\delta}$. Hence, we see that at $t = T$ the strategy of the firm $\pi_T$ is a homogenous function of degree 1, while the value of the firm is the homogenous function of degree $1 - \delta$. Next, we prove by induction that this result holds for an arbitrary $t$. First, we assume that the result holds at time $t$: $\pi_t = C_t k_t$ and $J(k_t, t; h) = B_t k_t^{1-\delta}$ and then we show that it also holds at time $t - 1$ and derive recursive relation between $C_t$ and $C_{t-1}$.

Assuming $\pi_t = C_t k_t$ and $J(k_t, t; h) = B_t k_t^{1-\delta}$ it is easy to see form (23), that time $t - 1$ problem looks as follows:

$$\max \left\{ \pi_{t-1}^{1-\delta} h + (1 - \lambda h) B_t C_t^{-\gamma(1-\delta)} \pi_{t-1}^{\gamma(1-\delta)} k_t^{(1-\gamma)(1-\delta)} \right\},$$

s.t. $k_t = (1 + \alpha h) k_{t-1} - \pi_{t-1} h$.

The first order condition with respect to $\pi_{t-1}$ for this equation, after some minor simplifications, is as follows:

$$h + (1 - \lambda h) B_t C_t^{-\gamma(1-\delta)} \pi_{t-1}^{1-(1-\gamma)(1-\delta)} k_t^{(1-\gamma)(1-\delta)} k_t^{(1-\gamma)(1-\delta)-1} \left[ \gamma \frac{k_t}{\pi_{t-1}} - h(1 - \gamma) \right] = 0,$$  

(41)

Now we show, that there exists homogenous solution to (41) which has the form $\pi_{t-1} = C_{t-1} k_{t-1}$. Taking into account that $k_t = (1 + \alpha h) k_{t-1} - \pi_{t-1} h = (1 + \alpha h - C_{t-1} h) k_{t-1}$ we rewrite
(41) as follows:
\[
h + (1 - \lambda h) B_t C_t^{-\gamma(1-\delta)} \left( \frac{1 + \alpha h}{C_{t-1}} - h \right)^{(1-\gamma)(1-\delta)} - \gamma = 0,
\]
As it can be easily observed form (42), \(C_{t-1}\) satisfying this equation depends only on time and the parameters of the model. It does not depend on endogenous quantities like capital at time \(t\).
Hence, we indeed have the homogenous solution \(\pi_{t-1} = C_{t-1}k_{t-1}\).

Let us denote the function on the left hand side of (42) as \(F(C_{t-1}, t)\). It can be easily observed, that \(F(C_{t-1}, t)\) is a monotonic function of \(C_{t-1}\).\(^3\) Moreover, \(F(C_{t-1}, t) \rightarrow -\infty\) as \(C_{t-1} \rightarrow (1 + \alpha h)/h\) and \(F(C_{t-1}, t) = h\) if \(C_{t-1} = \gamma(1 + \alpha h)/h\). This simple observation allows us to conclude that there exists unique solution to (42). It is easy to see, that \(C_{t-1} < (1 + \alpha h)/h\). On the other hand, it is also obvious that \(C_{t-1} > \gamma(1 + \alpha h)/h\), since otherwise the left-hand side of (42) is strictly positive. As a result, we get the following inequality for \(C_{t-1}\):
\[
\gamma \frac{1 + \alpha h}{h} < C_{t-1} < \frac{1 + \alpha h}{h}.
\]

Now it remains to show that our solution satisfies the second order condition. The second order condition for the problem (40) looks as follows:
\[
\pi_{t-1}^\gamma(1-\delta) - k_t (1-\gamma)(1-\delta) - 1 \left\{ \frac{k_t}{\pi_{t-1}} \right\} \left( \frac{k_t}{\pi_{t-1}} - h(1-\gamma) \right) \left( \gamma \frac{k_t}{\pi_{t-1}} - h(1-\gamma) \right) + \left[ -\gamma h - \gamma \frac{k_t}{\pi_{t-1}} \right] (1-\delta)(1-\lambda h) B_t C_t^{-\gamma(1-\delta)} - \delta(1-\delta) \pi_{t-1}^{-\delta-1} < 0
\]
It is now obvious that in order to prove that (44) holds it suffices to show that
\[
\left( \frac{k_t}{\pi_{t-1}} \right)^\gamma(1-\delta) - ((1-\gamma)(1-\delta) - 1)h \left[ \gamma \frac{k_t}{\pi_{t-1}} - h(1-\gamma) \right] < 0.
\]
Note, that for our homogenous solution \(\frac{k_t}{\pi_{t-1}} = \frac{1 + \alpha h}{C_{t-1}} - h\). Using this expression and inequality (43) we obtain
\[
\left[ \gamma \frac{k_t}{\pi_{t-1}} - h(1-\gamma) \right] = \gamma \frac{1 + \alpha h}{C_{t-1}} - h < 0.\]
In the same way we show that
\[
\frac{k_t}{\pi_{t-1}} \gamma(1-\delta) - ((1-\gamma)(1-\delta) - 1)h > -((1-\gamma)(1-\delta) - 1)h > 0.
\]
Hence, (45) holds. As a result, we see that second order condition holds.

Now we derive a recursive relation for \(C_{t-1}\). In order to do that, we derive the formula for \(B_t\). Since the value of the firm is \(J(k_t, t; h) = B_t k_t^{1-\delta}\), we notice from (23) that
\[
J(k_{t-1}, t-1; h) = B_{t-1} k_{t-1}^{1-\delta} = \pi_{t-1}^{1-\delta} h + (1 - \lambda h) B_t C_t^{-\gamma(1-\delta)} \pi_{t-1}^{1-\delta} k_t^{1-\delta} = \pi_{t-1}^{1-\delta} \left( h + (1 - \lambda h) B_t C_t^{-\gamma(1-\delta)} \frac{k_t}{\pi_{t-1}} \right)^{(1-\gamma)(1-\delta)}).
\]

\(^3\)This follows from the fact that \(F(C_{t-1}, t)\) can be expressed as a sum of two decreasing functions. To prove this observe that \(\gamma < 1\) and \((1 + \alpha h)/h - h)\((1-\gamma)(1-\delta)-1\)\(\gamma(1 + \alpha h)/h - h = \gamma \left( \frac{1 + \alpha h}{C_{t-1}} - h \right)^{(1-\gamma)(1-\delta)} + (\gamma - 1) h \left( \frac{1 + \alpha h}{C_{t-1}} - h \right)^{(1-\gamma)(1-\delta)-1}\)

\[
\gamma \left( \frac{1 + \alpha h}{C_{t-1}} - h \right)^{(1-\gamma)(1-\delta)} + (\gamma - 1) h \left( \frac{1 + \alpha h}{C_{t-1}} - h \right)^{(1-\gamma)(1-\delta)-1}.
\]

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From (41) we obtain that
\[
(1 - \lambda h)B_t C_t^{-\gamma(1-\delta)} \left( \frac{k_t}{\pi t-1} \right)^{(1-\gamma)(1-\delta)} = -h \frac{k_t}{\pi t-1} \left[ \gamma \frac{k_t}{\pi t-1} - h(1-\gamma) \right]
\]
Substituting this into expression for \(J(k_{t-1}, t-1; h)\) we obtain:
\[
B_{t-1}k_{t-1}^{1-\delta} = \pi_{t-1}^{1-\delta} h (1 - \gamma)k_t/\pi_{t-1} + h(1 - \gamma).
\]
Taking into account that \(\pi_{t-1} = C_{t-1}k_{t-1}\) and \(k_t = (1 + \alpha h)k_{t-1} - \pi_{t-1}h = (1 + \alpha h - C_{t-1}h)k_{t-1}\),
we obtain the following expression for \(B_t\):
\[
B_t = \frac{(1 - \gamma)(1 +\alpha h)C_{t-1}^{1-\delta}}{h - \gamma(1 + \alpha h)/C_t}.
\]
Substituting this expression in equation (42) we obtain:
\[
1 - (1 - \lambda h)(1 - \gamma)(1 + \alpha h) \frac{C_t^{(1-\gamma)(1-\delta) - 1}}{h - \gamma(1 + \alpha h)/C_t} \left( \frac{1 + \alpha h}{C_t} - h \right)^{(1-\gamma)(1-\delta) - 1} \left[ h - \gamma \frac{1 + \alpha h}{C_t} \right] = 0.
\]
Now we prove the second part of Proposition 4. We show, that \(J(k_t, t; h) \to 0\) as \(h \to 0\) if
\((1 - \lambda h)(1 - \gamma)(1 + \alpha h) < 1\).
\[
J(k_t, t; h) = \pi_t^{1-\delta} h + \frac{h}{\pi_t^{-\gamma(1-\delta)}} \sum_{\tau=t+1}^T (1 - \lambda h)^{\tau-t} \pi_t^{1-\delta}(1-\gamma).
\]
First, we show that the amount of the capital of the firm exponentially declines in the course of time. Indeed, \(k_t = (1 + \alpha h - C_{t-1})k_{t-1}\). Then, from (43) it follows that \(k_t < (1 + \alpha h)(1 - \gamma)k_{t-1} < \ldots < [(1 - \gamma)(1 + \alpha h)]^T k_0\). In what follows we denote \(\Lambda = (1 - \gamma)(1 + \alpha h)\). Taking into account that \(\pi_t = C_t^t/(1 + \alpha h)h/k_t < (1 + \alpha h)/h\Lambda^t\) we can easily obtain that \(J(k_t, t; h) < \tilde{\Lambda} h\delta\), where \(\tilde{\Lambda}\) is some constant. As a result, \(J(k_t, t; h) \to 0\) as \(h \to 0\).

Finally, we prove the third part of Proposition 4. To this end, we consider the following functions:
\[
f(C_t) = \left( h - \gamma \frac{1 + \alpha h}{C_t} \right) / C_t^{(1-\gamma)(1-\delta) - 1}
\]
and
\[
g(C_{t-1}) = (1 - \gamma)(1 - \lambda h)(1 + \alpha h) \left( \frac{1 + \alpha h}{C_{t-1}} - h \right)^{(1-\gamma)(1-\delta) - 1} \left[ h - \gamma \frac{1 + \alpha h}{C_{t-1}} \right].
\]
It can be easily verified that both functions are monotonically increasing, besides that, from (24) we conclude that \(f(C_t) = g(C_{t-1})\). Next step in our proof is to find such values \(C\), that \(f(C) = g(C)\). Finding these fixed points is tantamount to solving the following equation:
\[
\left( h - \gamma \frac{1 + \alpha h}{C} \right) / C^{(1-\gamma)(1-\delta) - 1} = (1 - \gamma)(1 - \lambda h)(1 + \alpha h) \left( \frac{1 + \alpha h}{C} - h \right)^{(1-\gamma)(1-\delta) - 1} \left[ h - \gamma \frac{1 + \alpha h}{C} \right].
\]
As it can be easily observed, this equation gives us two fixed points: \( C = \gamma (1 + \alpha h)/h \) and \( C = (1 + \alpha h - A)/h \), where \( A = ((1 - \lambda h)(1 + \alpha h)(1 - \gamma))/\tau (1 - \delta (1 - \gamma)) \). Denote

\[
C^* = \max \left\{ \gamma (1 + \alpha h)/h, (1 + \alpha h - ((1 - \lambda h)(1 + \alpha h)(1 - \gamma))/\tau (1 - \delta (1 - \gamma)) \right\}.
\]

The above formula gives the value of the fixed point which is the greater of the two. Clearly, there are no fixed points greater than \( C^* \). Moreover, if \( C > C^* \), then \( g(C) > f(C) \). This property follows from the monotonicity of functions \( f(x) \) and \( g(x) \), from the fact that there are no fixed points greater than \( C^* \) and from inequality \( g(\frac{1 + \alpha h}{h}) = +\infty > f((1 + \alpha h)/h) \). Consider now backward recursion \( f(C_t) = g(C_{t-1}) \) with initial value \( C_T = (1 + \alpha h)/h \), then it follows from Lemma 2 that \( C_t \to C^* \) as \( t \to -\infty \).

Now we are ready to prove the third part of Proposition 4. Suppose that \( h \) is fixed and \( T \) is sufficiently large. Hence, by virtue of convergence result, for sufficiently small \( t, C_t \) will be approximately equal to \( C^* \), since for high enough \( T, C_t \) will get close to \( C^* \). This means that for small \( t, k_t = (1 + \alpha h - C_{t-1})k_{t-1} \approx (1 + \alpha h - C^*)k_{t-1} \). Hence, if \( (1 + \alpha h - C^*) > 1 \) the firm will be accumulating capital for some period. As a result, the capital of the firm will be growing for some period of time. However, since \( k_T = 0 \), at some point this dynamics is reversed and the firm starts dissipating the capital, which gives rise to a hump shaped pattern of capital. To conclude the proof, we note that condition \( (1 + \alpha h - C^*) > 1 \) can be rewritten as

\[
1 + \alpha h - C^* h = \min \left\{ (1 - \gamma)(1 + \alpha h), ((1 - \lambda h)(1 + \alpha h)(1 - \gamma))/\tau (1 - \delta (1 - \gamma)) \right\} > 1.
\]

Finally, it is easy to see that this last inequality holds iff \( \Delta = (1 - \lambda h)(1 + \alpha h)(1 - \gamma) > 1 \).

Now it remains to show that the capital is either decreasing or a hump-shaped function of time. This assertion easily follows from Lemma 2 and the representation \( k_T = (1 + \alpha h - C_{T-1})h \). Indeed, it follows from Lemma 2 that \( C_1 < C_2 < \ldots < C_T \), hence, if \( (1 + \alpha h - C_1 h) > 1 \) there will be the growth of capital until time \( T \) such that \( (1 + \alpha h - C_T h) < 1 \) when the firm starts depleting its capital, and, as a result, function \( k_T = (1 + \alpha h - C_{T-1})h \ldots (1 + \alpha h - C_1 h) \) will have a hump-shaped pattern. If, however, \( (1 + \alpha h - C_1 h) < 1 \), then the function will be strictly decreasing.

**Proof of Proposition 5.**

Consider the Bellman equation for an oligopolistic firm (22). It is easy to see that in this recursive expression

\[
J_i(k_{it}, t; h) = \sum_{\tau = t}^{T} \pi_{it}^{1-\delta} \left( \pi_{ir}^{1-\delta} + y_{ir} \right)^{-\gamma} h.
\]

In the expression above, \( \pi_{ir}^{1-\delta} \) are functions of \( k_{it} \), which were chosen at time \( t = \tau \). Denote

\[
\tilde{J}_i(k_{it}, t; h) = J_i(k_{it}, t; h)(\pi_{it}^{1-\delta} + y_{it})^{-\gamma} = \sum_{\tau = t}^{T} \pi_{ir}^{1-\delta} \left( \pi_{ir}^{1-\delta} + y_{ir} \right)^{-\gamma} h.
\]

Proof of Proposition 5.

Consider the Bellman equation for an oligopolistic firm (22). It is easy to see that in this recursive expression
Then, the first order condition for problem (22) will be as follows:

\[(1 - \delta)\pi_{i(t-1)}^\delta h + (1 - \delta)\gamma\pi_{i(t-1)}^{\delta}((\pi_{i(t-1)}^{\delta} + y_{-i(t-1)})^{\gamma-1}J_i(k_{it}, t; h) - (\pi_{i(t-1)}^{\delta} + y_{-i(t-1)})^\gamma J_i(k_{it}, t; h)h = 0, \tag{48}\]

\[\tilde{J}_i(k_{it}, t; h) = \]

\[h(1 - \delta) \sum_{\tau=t}^{T} \left[ \pi_{ir}^{-\gamma} \left( \pi_{ir}^{1-\delta} + y_{-ir} \right)^{-\gamma} - \gamma \pi_{ir}^{1-\delta} \pi_{ir}^{-\gamma} \left( \pi_{ir}^{1-\delta} + y_{-ir} \right)^{-\gamma-1} \right] (\pi_{ir})_{ki} = \]

\[= h(1 - \delta) \sum_{\tau=t}^{T} \pi_{ir}^{-\gamma} \left( \pi_{ir}^{1-\delta} + y_{-ir} \right)^{-\gamma} \left[ 1 - \gamma \frac{\pi_{ir}^{1-\delta}}{\pi_{ir}^{1-\delta} + y_{-ir}} \right] (\pi_{ir})_{ki}'. \]

Now we prove that there exists symmetric equilibrium in which \(\pi_{ir} = \pi_{r} = C_r k_r\) where \(C_r\) depends only on time. To demonstrate this, we show that first order condition (48) is satisfied. First, consider formula (49). We note that if \(\pi_{ir} = C_r k_r\) then it easily follows from (21) that for any \(\tau > t, k_r = (1 + \alpha h - C_{\tau-1} h)(1 + \alpha h - C_{\tau} h)k_t = L_{\tau t} k_t\), where \(L_{\tau t}\) is a function of time only. As a result, \((\pi_{ir})_{ki} = C_r L_{\tau t} = \pi_{ir}/k_t\). Substituting \((\pi_{ir})_{ki} = \pi_{ir}/k_t\) and \(\pi_{ir} = C_r k_r\) into (46), (47) and (49) and taking into account that in a symmetric equilibrium \(\pi_{ir}^{1-\delta} + y_{-ir} = n\pi_{r}^{1-\delta}\), we obtain:

\[J(k_t, t; h) = B_t k_t^{1-\delta}, \quad \tilde{J}(k_t, t; h) = n^{-\gamma} B_t C_t^{-\gamma(1-\delta)} k_t^{(1-\delta)(1-\gamma)}, \]

\[\tilde{\tilde{J}}(k_t, t; h) = (1 - \delta) \left( 1 - \frac{\gamma}{n} \right) \tilde{J}(k_t, t; h)/k_t = \]

\[= (1 - \delta) \left( 1 - \frac{\gamma}{n} \right) n^{-\gamma} B_t C_t^{-\gamma(1-\delta)} k_t^{(1-\delta)(1-\gamma)-1}. \]

Note, that expressions above hold only in equilibrium. As a result, \(\tilde{\tilde{J}}(k_t, t; h)\) is not equal to the derivative of \(\tilde{J}(k_t, t; h) = n^{-\gamma} B_t k_t^{1-\delta} C_t^{-\gamma(1-\delta)} k_t^{(1-\delta)(1-\gamma)}\). Substituting the above expressions into (48) we get the following equation:

\[h + (1 - \lambda h) B_t C_t^{-\gamma(1-\delta)} \sum_{\tau=t}^{T} \left[ \gamma k_t / n \pi_{t-1}^{\delta} - h(1 - \gamma / n) \right] = 0. \tag{50}\]

Substituting now \(\pi_{t-1} = C_{t-1} k_{t-1}\) and \(k_t = (1 + \alpha h - C_t) k_{t-1}\) into (50) we get

\[h + (1 - \lambda h) B_t C_t^{-\gamma(1-\delta)} \left( \frac{1 + \alpha h}{C_{t-1}} - h \right)^{(1-\gamma)(1-\delta)-1} \left[ \gamma 1 + \alpha h / C_{t-1} - h \right] = 0. \tag{51}\]

Note, that the expression above is very similar to the expression (42) and therefore, the rest of the proof is exactly the same as in the monopolistic case. Here we just point out that it easily follows from (51) that \(\gamma/n(1 + \alpha h)/h < C_{t-1} < (1 + \alpha h)/h\) and hence, the solution does not exist for \(\gamma \geq n\).

\[Q.E.D.\]
References


