Mitigating Disruption Cascades in Supply Networks

Nitin Bakshi  
London Business School, Regent’s Park, London NW1 4SA, United Kingdom, nbakshi@london.edu

Shyam Mohan  
London Business School, Regent’s Park, London NW1 4SA, United Kingdom, smohan@london.edu

The losses from supply chain disruptions arise not only due to direct damage at firms, but also from the interruption of normal operations because of lack of supply; that is, due to disruption cascades from suppliers in the adjacent tiers and beyond. To curtail such losses, firms can make ex ante investments in mitigation and recovery strategies. In this paper, we use game-theoretic approach to study how firms’ investments depend upon network topology. We provide an exact characterization of each firm’s equilibrium investment and payoff. Strikingly, when the solution is interior, these quantities depend only on the attributes of the firm’s extended local neighborhood; that is, up to its tier-2 suppliers, thus making knowledge of the remaining network structure redundant. However, identifying the key firms in the network – firms that give the highest return-on-investment from a centralized perspective – requires a weighted centrality measure that depends on the entire network; knowledge of key firms facilitates opportunities for multilateral cooperation to (Pareto) improve welfare.

Key words: supply chain management; network games; cascades; disruption risk

History: May 5, 2016

1. Introduction

The interruption of normal operations at one or more firms in a supply chain can cascade through the network and wreak economic havoc. Examples of events that can trigger such cascades include natural disasters, labour strikes, bankruptcy filings, industrial accidents, and quality failures. Modern-day supply chains have proven to be particularly vulnerable to disruptions due to their global, interconnected and complex nature. For instance, the triple disaster of the earthquake/tsunami/nuclear-accident that struck Japan in 2011 resulted in economic losses estimated at $210 billion, of which only about $35 billion are insured losses. The disruption of the interconnected supply chains lasted for more than 6 months, and affected multiple industries such as automobile, electronics, steel, tire and rubber, chemicals, consumer goods, and even Disney theme parks [Airmic 2013].

We emphasize two features of disruption cascades in supply networks which serve to amplify the resulting economic damages. First, a noteworthy characteristic of disruption cascades is the ripple
effect: disruptions to a firm’s operations not only affect its immediate buyers, who cannot produce anymore, but based on the same principle often propagate along the supply chain to disrupt firms further away in the network. As per a survey by the Business Continuity Institute, nearly 40% of all supply-chain disruptions originate from the second tier and 10% of disruptions originate from beyond the second tier [Business Continuity Institute 2013]. Recent empirical evidence suggests that on average, idiosyncratic shocks that cause a 1% decline in revenues at the origin firm, also cause 0.82%, 1.03%, 0.87%, and 0.40% revenue decline in tier-1 to tier-4 connections, respectively [Wu 2016]. In the aftermath of the Japanese disaster, multiple instances of the ripple effect, due to single-sourcing of parts somewhere deep in the automobile supply chain, prompted this quote from Dave Andrea, senior Vice President of the Original Equipment Suppliers Association, “What vehicle manufacturers are finding are parts within parts... within parts that are sourced from a single-source [Japanese] manufacturer.” [Financial Times 2011].

Second, inherent to supply chains is the critical component property. It refers to the feature that the shortage of any one input from a set of complementary inputs, is enough to completely stall a production line; this is true regardless of the size of the supplier or the value of the input in short supply. An example which illustrates this property pertains to the disruption in manufacturing of Xirallic, a pigment used in metallic car paints [The Wall Street Journal 2011]. As of 2011, the only plant in the world producing Xirallic (belonging to the German firm, Merck KGaA) was located in Onahama in Japan, and this was severely damaged by the Tohuku earthquake and accompanying tsunami. Although a mere pigment for paint, Xirallic was an essential part of the bill-of-materials. Its shortage idled plants and consequently firms like Toyota Motor Corporation, Nissan, Ford and Chrysler faced a shortage of colors used in models that constituted up to 20% of their production volume. In general, the critical component property allows us to focus on suppliers with high input specificity (they are not easy to switch); empirically they are found to be the major contributors to disruption cascades [Barrot and Sauvagnat 2016 and Wu 2016].

The twin-features of the ripple effect and the critical component property form an economically lethal combination. However, firms can mitigate the risk from disruptions. On the one hand, firms can invest to reduce the probability of being directly disrupted by a trigger event; e.g., a manufacturing firm wanting to reduce the threat of fire-based damages to its plant, could invest in equipment maintenance, installation of fire alarms and sprinkler systems; or alternatively, the firm could also invest in spare capacity to which production can be shifted in the event of an unforeseen disaster [FM Global 2010]. On the other hand, firms can also invest in measures that counter the ripple effect; i.e., the likelihood of being disrupted by their suppliers. For example, firms can invest in inventory to buffer against temporary interruptions in supply, or they can identify avenues for alternate supply for a disrupted component [Tomlin 2006].
Thus, mitigating disruption cascades in supply networks involves developing long-term strategies for risk mitigation, and prioritizing the allocation of resources by “bang-for-buck”. Furthermore, firms have to make these strategic investment and resource allocation decisions well before the actual onset of a disaster. In the words of John Baranski, former Vice President of Smiths Medical, a leading supplier of medical equipment that undertook comprehensive risk mitigation in its supply chain, “We were eager to develop long-term risk mitigation strategies to support our business [...] We also wanted insight into... practical ways to prioritize our response and allocate our resources.” (FM Global 2011). However, managers investing in mitigation face some major hurdles; we highlight these below.

Supply networks often involve hundreds and thousand of interconnected firms spread across geographies. Figure 1 shows the buyer-supplier network in the U.S. as of 2006, and highlights the Big Three automakers. The global network is even more complex; for instance, Ford maintains more than 50 plants worldwide and has up to 10 tiers of suppliers, where tier 1 alone comprises 1,400 companies across 4,400 manufacturing sites (Simchi-Levi et al. 2015). It is evident that determining each firm’s equilibrium investment in mitigation, as a function of network structure, is a significant managerial challenge. Hence, our primary research focus in this paper is to establish the relationship between firm investment and network characteristics, when there is full visibility into the latter.

Figure 1 (Reproduced from Atalay et al. (2011)) The U.S. buyer-supplier network in 2006. GM, Ford, and Chrysler are in red, their suppliers are in orange, and other connected firms in gray.

A related concern is the ability to map out the entire network and to capture the characteristics and capabilities of all firms. In order to manage its supply chain risk better, in 2012 Toyota embarked on a major project to map out its entire supply chain network. The company quickly realized that more than half of its trusted supplier base was unwilling to provide visibility into their suppliers due to competitive reasons (Supply Chain Digest 2012). Similar challenges in mapping out
the network are faced by many other firms attempting to manage risk in their supply chains (Supply Chain Digest 2013). Hence, we are also keen to determine the informational burden associated with making the optimal investments in risk mitigation.

Finally, in addition to the aforementioned challenges, it is well-known that unilateral action can potentially lead to substantial inefficiency. Thus, an additional concern of ours is to develop an approach that helps identify and rank opportunities to overcome such inefficiency.

In order to capture the key supply-chain features in our setting, we create a stylized model of a given network of firms, where different suppliers to a firm supply complementary inputs. Firms make a one-time strategic decision regarding how much to invest to mitigate the probability of being disrupted either directly, or indirectly via disruption cascades. Investment decisions are determined in equilibrium as the solution to a game in which firms’ investments are best responses to the investments of their suppliers. We then analyze this model to shed light on our research focus described above.

Our main findings are as follows. First, we provide an exact characterization of a firm’s investment and payoff in the decentralized equilibrium. A striking feature of our results is that, when the firms make positive investments at all nodes and links (interior solution), the investments and payoffs typically depend only on properties of the extended local neighborhood, that is, up to the focal firm’s tier-2 suppliers. This limited dependence on network structure draws an interesting contrast with the extant literature which has highlighted that systematically important nodes are typically identified using global network metrics such as centrality (e.g., Acemoglu et al. 2013, Acemoglu et al. 2015). This contrast can be attributed to the fact that we consider endogenous investment in a game of strategic substitutes, whereby a firm’s suppliers invest to protect themselves from disruptions cascading down from their own suppliers, and thus, by virtue of the strategic substitutes property, essentially insulate the focal firm from risk posed by firms in higher tiers. On allowing for boundary solutions, we find that more information is needed but the intuition above remains relevant: Specifically, whenever a firm does not invest along a link to a tier-1 supplier, determining the focal firm’s investment requires information about the suppliers to the tier-1 supplier. A crucial implication of these findings is that unilateral risk mitigation typically requires knowledge only of the proximate neighborhood, supplemented with focused additional information in case of boundary solutions.

Second, as a possible means to reduce the inefficiency, we characterize the positive externalities induced by a firm’s investment on the payoff of other individual firms in the network, as well as

---

1 Inefficiency refers to the difference in aggregate supply-chain payoff achieved through centralized and decentralized decision making.

2 A global metric requires knowledge of the entire network structure.
on welfare. Such a characterization also helps identify key firms that give the highest return-on-investment from a centralized perspective. Thus, we offer a means to identify and rank opportunities for collaborative risk mitigation that lead to Pareto improvement, such as the joint investment by western retailers in improving the safety conditions at textile factories in Bangladesh in the aftermath of numerous safety incidents (Bloomberg 2013). The characterization of externalities requires full network information, as opposed to the “limited dependence” found in the context of unilateral or decentralized decision making.

Finally, we are able to relate the inefficiency to the network structure, in particular, to the distribution of the in-degree and the out-degree for networks. Restricting attention to the interior solution, we find that inefficiency tends to be higher when the distribution for in-degree is such that more assembly stars (e.g., one assembler, multiple suppliers) are likely to exist in the network; the inefficiency tends to be lower when the distribution of out-degree is such that more distribution stars (e.g., one warehouse, multiple retailers) are likely to exist; and the inefficiency is intermediate relative to the previous two cases for networks with more degree balance, e.g., linear network.

As such, we provide guidance regarding which network topologies stand to benefit the most from collaborative interventions in risk mitigation.

2. Literature Survey

Our work is related to three streams of literature - management of disruption risk in supply chain networks; games on networks; and contagion in financial and economic networks. We discuss each in turn.

2.1. Disruption risk in supply chain networks

The importance of studying disruptions to supply-chain networks and their impact on business has been highlighted qualitatively in Kleindorfer and Saad (2005) and Netessine (2009). The former provides a conceptual framework for understanding the general area, while the latter highlights the need for new approaches that support a network-based view of supply chain management. Subsequently, a few papers have adopted a network perspective for the problem of supply chain disruptions. Bimpikis et al. (2015) characterize the supply equilibrium (in terms of price and quantity decisions) to rank network structures in terms of the profit, welfare and the consumer surplus that they generate. Ang et al. (2016) and Bimpikis et al. (2014) use a Principal-Agent framework to show how the problem of moral hazard leads to suboptimal configurations in multi-tier supply chains. DeCroix (2013) proposes heuristic solutions for determining the optimal inventory policy

3 The in-degree of a firm corresponds to the number of suppliers it has, while the out-degree of a firm corresponds to the number of buyers it has.

4 Star networks are also sometimes referred to as hub-and-spoke or core-periphery networks.
for a general assembly system facing disruption risk. However, the above papers are quite limited in terms of the type of network topologies they can handle, either because they have a different focus or due to tractability issues.

To get around the problem of tractability, a few papers have used simulation and computational techniques to tackle the problem of disruption cascades. For instance, Kim et al. (2015) use a simulation-based approach to relate network structure to resilience and find that a power-law distribution of degrees gives rise to the most resilient topologies. In a recent work, Simchi-Levi et al. (2015) adopt a computational approach to study risk mitigation for Ford’s internal supply chain: Given a particular node is disrupted for a length of time equal to its TTR (time to recover), the authors provide linear programs to numerically determine the production quantities and inventories to be held at different supplier locations that minimize the economic impact of the disruption. In contrast, we analytically study disruption cascades in inter-firm (external) supply chains, wherein firms make long-term strategic choices pertaining to mitigation and recovery capability.

Recently, a few empirical papers have also documented the significance and severity of disruption cascades in supply chains, most notably, Barrot and Sauvagnat (2016), Wu (2016) and Wang et al. (2016). A related idea is explored in Osadchiy et al. (2015) who empirically study how the supply chain network structure relates to propagation of systematic risk; the latter is defined as the correlation coefficient of sales change with market return.

Within operations management, the phenomenon of disruption cascades in networks is also studied in the literature on reliability of complex networked systems; see for example Barlow and Proschan (1996). A recent example from this literature is Kim and Tomlin (2013) who develop a game-theoretic model for capturing risk mitigation and loss sharing between subsystems, under investments in failure prevention and recovery capacity. A key differentiating feature of this setting is that either the entire system fails or it does not, whereas in supply disruption cascades not all firms are necessarily affected: losses may be limited to a few firms that are affected due to the ripple effect.

2.2. Games on networks

The literature on games on networks is vast and comprises papers published over the last 20 years. For a detailed summary of this area, we refer the reader to Jackson (2010), Goyal (2012) and Jackson and Zenou (2014). Games on networks have been studied in a number of different contexts. One way to broadly classify such games is as either games involving strategic complements, or as games involving strategic substitutes, based on whether the marginal utility of a player’s effort increases or decreases, respectively, when neighbors increase their effort. Our game of endogenous investment to mitigate disruption risk demonstrates the property of strategic substitutes, as increased investment
from neighboring firms encourages free-riding behaviour for a firm. A well-studied problem in economics exhibiting strategic substitutes is that of provision of public goods to a network of individuals (Elliott and Golub (2013), Bramoullé and Kranton (2007), Bramoullé et al. (2014)). A related theme is that of risk sharing using mutual insurance amongst a population of individuals (Bloch et al. (2008)).

While most studies of economic networks assume that all firms have complete knowledge of the network, this assumption is relaxed in Galeotti et al. (2010) where individuals know only about themselves and have a belief over the degrees of their neighbors. The authors study equilibrium actions under strategic substitutes and complements and draw the connection between network topology and equilibrium actions. de Martí and Zenou (2013) also characterize Bayesian Nash equilibrium in a network game of strategic complementarities and relate the centralities to the efforts of agents.

The focus in this literature is not on disruptions and how they cascade through a network, hence, the modeling approach and resulting insights are quite distinct from ours.

2.3. Contagion in financial and economic networks

Another problem that has characteristics that are similar to supply chain disruptions is contagion in financial networks. Although the problem of contagion had been identified earlier (e.g., Eisenberg and Noe (2001), post the financial crisis of 2008, several papers in the literature study the relationship between network structure, financial contagion and vulnerability of banks. Elliott et al. (2014) brings out the trade-offs between integration (increased dependence on counter-parties) and diversification (increased number of counter-parties per bank) and the counter-balancing effects of integration and diversification on cascading defaults in a financial network. Acemoglu et al. (2015) propose a model which sheds light on the robust-yet-fragile property of financial networks. When shocks to assets are small in magnitude, a connected network enhances stability and mitigates default risk; however, as shocks increase in magnitude, connections serve as channels of propagation of shocks and defaults through the financial network. The authors then characterize network structure that ensure stability of the financial system. Supply chain disruptions differ from financial contagion in two ways. First, in supply chains, due to bill-of-material considerations, firms cannot choose to source from or supply to any arbitrary firm in the network simply because it reduces their disruption probabilities. Such a realignment is possible in the case of interbank lending among banks. Second, in financial networks there is typically no parallel to link-based investments (aimed to prevent propagation of disruptions) that are common in supply chain networks.

A notable exception in this regard is Zawadowski (2013) which models an entangled financial system in which an individual bank may invest in counterparty insurance to prevent the cascading of failures. However, the analysis therein is still distinct from our work due to the absence of bill-of-material considerations.
Finally, it is worth noting an influential strand in the economics literature that uses a passive (non game-theoretic) approach, in contrast to our strategic approach, to study how intersectoral linkages in an economy can amplify and cascade productivity shocks (e.g., Acemoglu et al. 2012, Gabaix 2011).

3. Model Description
3.1. Graph
We consider a supply chain network comprising \( N \) firms represented by a directed acyclic graph \( G(V,E) \); the firms are nodes in \( V \) and there is an edge \( i \rightarrow j \in E \) whenever firm \( i \) is a supplier to firm \( j \). Let \( N_i \) be the set of suppliers to firm \( i \), with cardinality of the set denoted by \( |N_i| \). The \( |N_i| \) suppliers supply complementary components to firm \( i \); i.e., firm \( i \) might potentially be disrupted (unable to produce and meet the demand of its buyers) even if just one of its \( |N_i| \) suppliers is disrupted.

3.2. Probability of Disruptions
Sample Space: We consider a one-shot setting in which firms could be disrupted. We say a firm is disrupted when it is unable to carry its usual operations, either due to an external disruption in its facilities (idiosyncratic disruption) or due to unavailability of adequate supply (cascade from suppliers). To describe the probabilistic nature of such disruption cascades on \( G \), we consider a sample space \( \Omega \), wherein a single outcome \( \omega \in \Omega \) contains information on the list of disrupted firms, and their mode of disruption, i.e., idiosyncratic or cascade, and the path of the disruption, in the case of a cascade, see Figure 2).

For tractability and in keeping with our setting of complementary suppliers, we make the following assumption to limit the sample space \( \Omega \): in any outcome, a firm, if disrupted, is disrupted by at most one mode of disruption, i.e., either via an external disruption, or through a cascade from exactly one of its suppliers. We believe this assumption on the sample space is not overly restrictive, as the firm’s losses remain the same regardless of how many suppliers cascade their disruption to it, or whether it had an external disruption. Corresponding to this assumption, note that in Figure 2, though both firms 1 and 2 are disrupted in the sample outcome \( \omega \), only firm 1 has actually disrupted firm 3 (shown by a red link between 1 and 3).

Probability Space: To complete the formal description of the probability space, we embed the sample space in the usual probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( \mathcal{F} \), the sigma-algebra is the set of all measurable events, and \( \mathbb{P} \) specifies the probability measure assigned to the events in \( \mathcal{F} \). Since

\( ^6 \) Equivalently, this very outcome space can alternately be derived from a more detailed dynamic description, where disruptions to a node are modeled as continuous-time arrival processes from disrupted suppliers or external sources. Making the standard assumption that two arrivals do not occur simultaneously, in any sample path, a firm’s disruption can then be attributed exclusively to the first arrival.
the sample space is countable in our model, we set the sigma algebra to be the power set of the sample space, i.e., $\mathcal{F} = 2^\Omega$. Filtering and labeling the outcomes of interest from the perspective of a focal firm $i$, let $\Omega_i \subseteq \Omega$ be the set of all outcomes in which firm $i$ is disrupted. The set $\Omega_i$ can be partitioned into smaller subsets, based on how $i$ was disrupted. Let $\mathcal{D}_i \subseteq \Omega_i$ be the subset of outcomes in which firm $i$ was disrupted due to an idiosyncratic disruption. Let $\Omega_{ji} \subseteq \Omega_i$ be the set of outcomes where firm $i$ was disrupted due to a cascade from disruption to firm $j$, $j \in N_i$. As $j$ has to be disrupted in the outcomes in the set $\Omega_{ji}$, clearly $\Omega_{ji} \subseteq \Omega_j$ holds true as well.

Assigning probability measures to the different outcomes of interest, define by $w_i := \mathbb{P}(\Omega_i)$ the total probability of disruption of firm $i$. Let $\theta^0_i := \mathbb{P}(\mathcal{D}_i)$ be the probability of an external disruption at $i$. Let $\theta_{ji}$ be the conditional probability of $i$ being disrupted given firm $j$ is disrupted (cascade from $j$ to $i$). This conditional probability can be expressed as a function of the probabilities of the individual outcomes as: $\theta_{ji} := \frac{\mathbb{P}(\Omega_{ji})}{\mathbb{P}(\Omega_j)}$, wherein $\Omega_{ji} \subseteq \Omega_j$. Our assumption on at most one mode of disruption for a firm corresponds to: $\mathcal{D}_i \cap \Omega_{ji'} = \emptyset$ and $\Omega_{j'i} \cap \Omega_{j'i'} = \emptyset$ for any $j', j'' \in N_i$. Hence,

$$w_i = \mathbb{P}(\Omega_i) = \mathbb{P}(\mathcal{D}_i \cup (\cup_{j \in N_i} \Omega_{ji})) = \theta^0_i + \sum_{j \in N_i} \theta_{ji} w_j \quad (1)$$

To illustrate our modeling approach, in Figure 3 we consider the example of a simple two-node supply chain with $S$ as the supplier and $B$ as the buyer, and list out the five mutually exclusive and collectively exhaustive outcomes.

Outcome $\omega_1$ describes the scenario where firm $S$ is externally disrupted, and firm $B$ is undisrupted. Outcome $\omega_2$ is the converse where $B$ alone is disrupted externally. Outcome $\omega_3$ is one where both firms are disrupted by external disruptions. This could happen if, for example, $S$ and $B$ were collocated, and were affected by the same natural disaster. In outcome $\omega_4$, $S$ is externally disrupted, and the disruption cascades from $S$ to $B$. For this to happen, $B$ must not be disrupted externally. The failure of $B$ is completely due to the inability of $S$ to meet demand. It is clear that outcomes $\omega_1$ to $\omega_5$ are mutually disjoint and collectively exhaustive. The following equations relate these probability measures to the probabilities of the outcomes described in Figure 3.
<table>
<thead>
<tr>
<th>Outcome</th>
<th>Description</th>
<th>Disruptions in the outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>Firm $S$ is externally disrupted. Firm $B$ is undisrupted.</td>
<td>$S \rightarrow B$</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>Firm $B$ is externally disrupted. Firm $S$ is undisrupted.</td>
<td>$S \rightarrow B$</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>Both firms $S$ and $B$ externally disrupted.</td>
<td>$S \rightarrow B$</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>Firm $S$ is externally disrupted, and then firm $B$ is disrupted due to a cascade from firm $S$.</td>
<td>$S \rightarrow \ldots \rightarrow B$</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>Neither firm is disrupted.</td>
<td>$S \rightarrow B$</td>
</tr>
</tbody>
</table>

**Figure 3** Example: Potential outcomes in a two-node supply chain setting.

\[
\begin{align*}
\theta_S^0 &= \mathbb{P}(\omega_1) + \mathbb{P}(\omega_3) + \mathbb{P}(\omega_4) \\
\theta_B^0 &= \mathbb{P}(\omega_2) + \mathbb{P}(\omega_3) \\
\theta_{SB} &= \frac{\mathbb{P}(\omega_4)}{\mathbb{P}(\omega_1) + \mathbb{P}(\omega_3) + \mathbb{P}(\omega_4)} \\
w_S &= \mathbb{P}(\omega_1) + \mathbb{P}(\omega_3) + \mathbb{P}(\omega_4) = \theta_S^0 \\
w_B &= \mathbb{P}(\omega_2) + \mathbb{P}(\omega_3) + \mathbb{P}(\omega_4) \\
&= \theta_B^0 + \theta_{SB} \theta_S^0
\end{align*}
\]

### 3.3. Investment in risk mitigation

In this subsection, we seek to endogenize the probabilities of disruption by expressing them as a function of investment made by firms with a view to minimizing expected losses. A firm $i$ allots an amount $y_i$ to reduce expected losses in its supply chain. This investment $y_i$ is divided into $|N_i| + 1$ parts: $y_i^0$ and $y_{ji}$, $j \in N_i$. The node investment $y_i^0$ is made with a view to reducing the firm's external disruption risks. Examples of such investment include equipment maintenance, and installation of fire alarms and sprinkler systems, hardening of buildings against damage from earthquakes or floods, reserving spare production capacity, etc. The link investment $y_{ji}$ is targeted to minimize the risk of cascading disruptions from the suppliers. Holding inventory, and maintaining the capability to quickly identify alternate suppliers with spare capacity are often mentioned examples of such operational measures (e.g., [PR Newswire 2000](#), [Reuters 2012](#), [Tomlin 2006](#)). Our abstraction, wherein we represent operational strategies, such as holding inventory and making contingent supply arrangements, simply as investments helps us formulate a parsimonious model that allows us to focus on network effects in supply disruption cascades.
In the light of the set of outcomes in Figure 3, consider the impact of investments \( y^0_S \), \( y_{SB} \) and \( y^0_B \). Each of these investments create a shift of the existing probability measures on the outcomes \( \omega_i \), \( i = 1, 2, \ldots, 5 \), and such a shift changes the effective probabilities of disruption, \( w_S \) and \( w_B \). We shall examine the impact of each of the three investments in turn.

- \( y^0_S \) causes a reduction in the measure assigned \( \omega_1 \) and this measure is transferred to outcome \( \omega_5 \). Similarly, there is a reduction in \( C \) and this mass shifts to \( \omega_2 \). Thus, the investment creates a reduction in the measure of \( w_S \) and no change in \( w_B \).

- Investment \( y_{SB} \) causes a reduction in the measure of \( \omega_4 \), and the measure is shifted to \( \omega_1 \). Thus for firm \( B \), there is a reduction in the cascade probability term \( \theta^0_S \theta_{SB} \), while the terms \( \theta^0_B \) and \( w_S \) are unaffected.

- Investment \( y^0_B \) shifts measure from \( \omega_2 \) to \( \omega_5 \) and \( \omega_3 \) to \( \omega_1 \). Thus there is a reduction in \( \theta^0_B \) and consequently in \( w_B \), and the remaining terms are unchanged.

We denote by \( p^0_i(y^0_i) \) and \( p_{ji}(y_{ji}) \), respectively, the probability of external disruption to firm \( i \), and the conditional probability of firm \( j \) disrupting \( i \), given firm \( j \) is disrupted. We assume an exponential dependence between the investments and probabilities, i.e., \( p^0_i(y_i) = \theta^0_i e^{-y^0_i/\alpha^0_i} \), and \( p_{ji} = \theta_{ji} e^{-y_{ji}/\alpha_{ji}} \), where \( \alpha^0_i, \alpha_{ji} > 0 \forall i, j \). Besides being tractable, the exponential form is useful for two reasons: one, it captures a monotonically decreasing trend of probability with investments, and two, it exhibits a decreasing marginal reduction in probability with increasing investment.\(^7\)

To obtain a given disruption probability, a link with a greater \( \alpha_{ji} \) requires a greater investment. For a fixed level of investment on link \((j, i)\), higher \( \alpha_{ji} \) is indicative of a greater conditional probability of firm \( i \) being disrupted given firm \( j \) is disrupted. Such a link property is often a function of the characteristics of both firms: \( i \) as well as \( j \). To illustrate, the difficulty of finding alternate supply may be a function of not only the type of component in question (a property of firm \( i \)'s bill-of-materials), but also a function of the willingness of firm \( j \) to share its expertise/technology in manufacturing this component with potential replacement suppliers. We refer to \( \alpha \) as inverse sensitivity, which captures its role in mediating the impact of investment on disruption probability.

Observe that \( \alpha \) for a particular node or link is equal to the amount of investment necessary to bring down the disruption probability across the node or link by about 63% (= 1 - 1/e).

We have made the reasonable assumption that an investment in a specific node or link exclusively mitigates the disruption probability associated with the respective node or link. In Appendix B we discuss a more general model where investments in one link can potentially affect the probability.

\(^7\) This specific functional form also relates to logistic regression (see, for example, Bakshi and Kleindorfer 2009 and Greene 2008), wherein the probability of disruption \( p \) is modeled as a function of the investment level \( y \), 
\[
\log \left( \frac{p}{1-p} \right) = a - \frac{y}{\alpha} \]
where \( a \) and \( \alpha \) are non-negative constants. For \( p \ll 1 \), as is the case for rare events, the logistic regression model corresponds to the exponential functional form which we use.
of disruption along another link; we find that although such a formulation imposes significant expositional and analytical burden, our qualitative insights remain unchanged.

Based on these inverse sensitivities, we define measures of weighted in-degree and out-degree for firms in our model. Many useful results in the later sections of this paper will be expressed in terms of these quantities.

**Definition 1 (Weighted in-degree).** For any firm $i$, the weighted in-degree $d^-_i$ is represented as the sum of its node-specific inverse sensitivity and the inverse sensitivities of all its incoming links, i.e., $d^-_i = \alpha^0_i + \sum_{j \in N_i} \alpha_{ji}$.

**Definition 2 (Weighted out-degree).** For any firm $i$, the weighted out-degree $d^+_i$ is the sum of inverse sensitivities of all its outgoing links, i.e., $d^+_i = \sum_{j \in V} \alpha_{ij} 1_{i \in N_j}$.

Note that we use weighted in-degree to include not only the inverse sensitivities of incoming links to $i$, but also the inverse sensitivity of node $i$ itself. The latter corresponds to the responsiveness of external disruption probability of firm $i$ to investment $y^0_i$. We also assume that, for any firm, the parameters $\theta$ and $\alpha$ and the functional form of the dependence of probabilities on investments are common knowledge.

### 3.4. Losses for firms

We denote by $l_i$ the loss incurred by firm $i$ in all outcomes $\omega \in \Omega_i$ where it is disrupted; hence, $l_i \omega_i$ is the expected loss for firm $i$. The loss for firms arising as a result of supply chain disruptions can be attributed to a number of causes, e.g., loss in productivity, increased cost of working, lost sales, loss in reputation and market share, or the fall in share value following the disruption [Business Continuity Institute 2012, Hendricks and Singhal 2005].

With the model details now in place, we present our results in the subsequent sections. We divide our results into three sections: in §4, we analyze the investment game in a decentralized setting. In §5 we provide a mathematical characterization of positive externalities from a firm’s investments. We then build on the results from §4 and §5 in §6 to analyze the centralized benchmark, and examine the dependence of inefficiency on the network structure. Although, for simplicity, in the main model we focus on complementary inputs and the interior solution, in §6 we discuss extensions that address these assumptions.

### 4. Decentralized decision-making

In this section, we shall consider firms making decentralized decisions regarding optimal investments for risk mitigation. Firms are faced with two questions pertaining to investments: firstly, what is the optimal amount of total investment, given the probabilities of disruption; and secondly, how to allocate this investment across the node and all incoming links, with a view to protecting itself from the various possible sources of disruption. The optimal investment for a firm could
potentially depend on its losses in the event of a disruption, the inverse sensitivity of the links from its suppliers, and its position in the overall network, and the firm needs to factor in all of the above in its optimization problem.

Given firm $i$’s total investment, $y_i$, we first address the problem of optimal allocation of this investment using the following program. Here the disruption probability of firm $i$, $w_i$, is a function of the corresponding probabilities $w_j$ for firms $j \in N_i$ and $y_i$. We note that the following problem is equivalent to minimizing the expected losses $l_iw_i$ with an investment budget $y_i$.

\[
\begin{align*}
 w_i(y_i, w_j, j \in N_i) &= \min_{y_i^0 + \sum_{j \in N_i} y_{ji} = y_i} \theta_i^0 e^{-y_i^0/\alpha_i^0} + \sum_{j \in N_i} w_j \theta_{ji} e^{-y_{ji}/\alpha_{ji}} \\
 \text{(ALLOC)}
\end{align*}
\]

For simplicity in exposition, going forward, we focus on the interior solution to the above optimization problem, i.e., when there is non-zero investment (at least 1 pence) on all nodes and links, and discuss the boundary cases in Section 7. The interior solution to ALLOC is characterized in the following lemma.

**Lemma 1 (Optimal allocation).** For a given investment level $y_i > 0$, the necessary and sufficient conditions for an interior solution to the allocation problem, in which $y_i^0 > 0$ and $y_{ji} > 0$, $\forall j \in N_i$ are respectively given by:

\[
\begin{align*}
 y_i > d_i^- \log \left( \frac{\alpha_i^0}{\theta_i^0} \right) - \alpha_i^0 \log \left( \frac{\alpha_i^0}{\theta_i^0} \right) - \sum_{j \in N_i} \alpha_{ji} \log \left( \frac{\alpha_{ji}}{w_j \theta_{ji}} \right), \\
 y_i > d_i^- \log \left( \frac{\alpha_{ji}}{w_j \theta_{ji}} \right) - \alpha_i^0 \log \left( \frac{\alpha_i^0}{\theta_i^0} \right) - \sum_{j \in N_i} \alpha_{ji} \log \left( \frac{\alpha_{ji}}{w_j \theta_{ji}} \right). \\
\end{align*}
\]

The interior solution is unique, and the corresponding disruption probability of firm $i$ is a continuous and convex function in $y_i$ and is given by the following expression:

\[
\begin{align*}
 w_i(y_i, w_j, j \in N_i) &= d_i^- \exp \left( -\frac{y_i}{d_i^-} \right) \left( \frac{\theta_i^0}{\alpha_i^0} \right) \frac{\alpha_i^0}{d_i^-} \prod_{j \in N_i} \left( \frac{w_j \theta_{ji}}{\alpha_{ji}} \right) \frac{\alpha_{ji}}{d_i^-}. \\
\end{align*}
\]

Clearly, ALLOC is a convex program in the investments $(y_i^0, y_{ji}, j \in N_i)$, where the total budget of firm $i$, $y_i$, is allocated towards reducing both idiosyncratic risk of firm $i$ and the cascading risk from neighboring firms. Using (3), we observe that, at the interior solution, the disruption probability $w_i$ is exponentially decreasing in the investment $y_i$, the rate of the exponent being the inverse of the weighted in-degree, $1/d_i^-$. It can also be shown that $w_i$ is non-decreasing in the investments $y_i^0$ and $y_{ji}$.\[8\] For convenience in exposition, we abuse notation slightly to represent the probability of disruption for firm $i$, both before and after optimal allocation, by $w_i$. Similar notation scheme is used for the investments $y_i^0$ and $y_{ji}$.
parameters $\theta^0_i, \alpha^0_i, \theta^i_j$, and $\alpha^i_j$, $j \in N_i$. Given the solution to the implicit equation (3), firm $i$’s problem reduces to the determination of the optimal total investment $y_i$. Firm $i$’s payoff in such a case can be written as follows.

$$\max_{y_i} U_i(y_i) = -l_i w_i - y_i \text{ s.t. (2) and (3) hold.} \quad \text{(DECEN)}$$

We are interested in analyzing the solution to the game $G(V, \{y_i\}, \{U_i\}, i \in V)$, where the firms are players, and each firm $i \in V$ chooses an investment $y^*_i$ to maximize its payoff, and the investment is the best response, given the investments of its suppliers. The solution to the decentralized problem is as follows.

**Proposition 1 (Solution to decentralized problem).** A unique interior solution is achieved when the loss $l_i$ is greater than a threshold, i.e., $l_i > \max \left( \frac{\alpha^0_i}{\theta^0_i}, \max_{j \in N_i} \frac{l_j \alpha^i_j}{\theta^i_j d^*_j} \right)$, and this solution can be expressed as follows:

- **Optimal investment of firm $i$** is given by:

  $$y^*_i(w^*_j, j \in N_i) = d_i^* \log(l_i) - \sum_{j \in N_i} \alpha^i_j \log \left( \frac{\alpha^i_j}{w^*_j \theta^i_j} \right) - \alpha^0_i \log \left( \frac{\alpha^0_i}{\theta^0_i} \right), \quad (4)$$

  where the equilibrium disruption probability $w^*_i$ for firm $i$ is:

  $$w^*_i(w^*_j, j \in N_i) = d_i^* / l_i. \quad (5)$$

- **The link and node disruption probabilities at equilibrium are:** $p^*_j = \frac{\alpha^i_j}{l_i w^*_j}$ and $p^0_i = \frac{\alpha^0_i}{l_i}.$

Proposition 1 characterizes the equilibrium investments and the disruption probabilities in the interior solution, which is ensured (using (2), (4), and (5)) if the loss $l_i$ is greater than a threshold that depends only on model primitives: $l_i > \max \left( \frac{\alpha^0_i}{\theta^0_i}, \max_{j \in N_i} \frac{l_j \alpha^i_j}{\theta^i_j d^*_j} \right)$, where $w^*_j = d^*_j / l_j$. We find that the equilibrium investments are proportional to the firm’s own losses, but inversely proportional to the suppliers’ losses. That is, if a firm sources from suppliers that face severe losses in the event of a disruption, then the firm’s own incentive to invest in risk mitigation is reduced, as such suppliers are bound to invest more themselves. Also, we note that a firm’s equilibrium investment increases both with its own in-degree and with its suppliers’ in-degree.

We make two crucial observations about the equilibrium outcome. First, we note that the equilibrium investments of a firm depend only on the properties of its extended local neighborhood, i.e., up to its tier-2 suppliers. This observation is in contrast to much of the existing literature on networks in which measures such as eigenvector centrality, bottleneck centrality (Acemoglu et al. 2013) or harmonic distance (Acemoglu et al. 2015) determine, in some sense, the systemically important nodes which merit higher investments. These measures are global, as their computation
entails knowledge of the entire network structure. By comparison, we find that firm $i$'s investment $y_i^*$ depends on the losses and inverse sensitivities of firm $i$ and its suppliers; suppliers’ inverse sensitivity calls for knowledge of second-tier suppliers (as explained in §3.3). This turns out to be the case because of the following reason. Since investments are endogenous in our setting, and the supply chain networks we consider are directed and acyclic (therefore disruptions cascade from an initially disrupted firm down to its buyers, and so on), at equilibrium, it suffices for firm $i$ to only consider the disruption probabilities of its immediate suppliers, as these suppliers would have accounted for the risk due to disruption cascades from higher tiers into their own optimal investment decisions, and thereby largely shielded the focal firm $i$ from these risks.

Second, from (5) we see that the expected losses $l_i w_i$ for firm $i$ is equal to its weighted in-degree. Regardless of a firm’s initial disruption probabilities or its “global” position in the network or the investment decisions of its suppliers, the expected losses depend only on a firm-specific model primitive.

Equation (5) also helps us get a handle on the relationship between supply-chain resilience and network structure. Since we define resilience of the supply chain as the negative of the total expected losses, resilience turns out to be the negative of the sum of weighted in-degrees of all firms in the network. This result conforms to the intuition that, as supply chains become more interconnected (the bill-of-materials of firms becomes more complex), they become less resilient. However, recall that our model has considered only complementary inputs thus far. When connectivity increases by the addition of more substitutes, it will reduce the total expected losses and enhance resilience. We discuss the model in the presence of substitutes in §7.

The two observations together imply that a firm’s equilibrium payoff depends only the properties of its extended local neighborhood (up to tier 2). This has important implications for practice. As discussed in Section 1, a major hurdle to effective supply risk management is the ability to map out the entire network and to capture the characteristics and capabilities of all firms in it. In 2012, when Toyota tried to map its supplier network, it faced push-back from more than half of the firms in its supplier base; they were unwilling to provide visibility into their suppliers for competitive reasons (Supply Chain Digest 2012). A more recent update suggests that Toyota has been able to identify 75% of its tier-2 suppliers and 40% of tier-3 vendors through an online census (Automotive News 2014). Our results show that, if losses and inverse sensitivities of suppliers are known precisely, it is not necessary to know the entire network to make equilibrium investment decisions. However, network information may still be quite useful if the system is not in equilibrium, or to improve beyond the decentralized equilibrium, as we illustrate in the next subsection.
5. Characterizing Positive Externalities

In this section, we build on the results of the §4 (specifically Lemma 1), to study network externalities; specifically, the dependence of disruption probabilities of one firm on the investments of other firms in the network. As such, the investments made by firms in the decentralized equilibrium ignore the externalities they impose on the rest of the network, and are therefore weakly inefficient. It is useful for a firm to be able to identify other firms in their extended supply network who under-invest, and thereby have a marked impact of the focal firm’s risk exposure. A firm that is aware of such network dependencies can then pursue opportunities for collaborative risk mitigation (e.g., subsidize the investments made by “problematic” network partners) which can result in mutual improvement.

The following lemma is a technical result that serves two ends. Firstly, it captures the posterior conditional probabilities of disruption, i.e., given a firm is disrupted, the probability that a particular supplier cascaded the disruption. These posterior probabilities turn out to be a straightforward function of the inverse sensitivities. Secondly, it provides matrices $A$ and $B$ that capture network relationships; we use $B$ to subsequently characterize externalities.

**Lemma 2 (Posterior probabilities).** Given firm $i$ is disrupted, the conditional probability that it has been disrupted by firm $j$, where $j \in N_i$, is $a_{ij} := \frac{\alpha_{ji}}{d_i}$. Further, if $A = [a_{ij}]$ and $a_{ii} = 0$ $\forall i$, the matrix $B = (I - A)^{-1}$ exists and is unique. Moreover, the $(i,j)^{th}$ entry in $B$ is the sum, over paths from $j$ to $i$, of conditional probabilities that disruption has cascaded from $j$ to $i$ over these paths, given firm $i$ is disrupted.

In the following example, we illustrate the implication of Lemma 2 for a simple linear network with $N$ nodes.

**Example 1 (Linear network).** We assume the nodes and links are homogeneous, i.e., $\theta_i^0 = \theta_n$ and $\alpha_i^0 = \alpha_n \forall i$, $\theta_{ji} = \theta_l$ and $\alpha_{ji} = \alpha_l \forall j \to i \in E$. The $A$ and $B$ matrices are as follows.

$$ A_{linear} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \frac{\alpha_l}{\alpha_l + \alpha_n} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\alpha_l}{\alpha_l + \alpha_n} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\alpha_l}{\alpha_l + \alpha_n} & 0 \end{bmatrix}; \quad B_{linear} = \begin{bmatrix} \frac{1}{\alpha_l} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\alpha_l}{\alpha_l + \alpha_n} & 0 & \cdots & 0 \\ \left(\frac{\alpha_l}{\alpha_l + \alpha_n}\right)^2 & \frac{\alpha_l}{\alpha_l + \alpha_n} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \left(\frac{\alpha_l}{\alpha_l + \alpha_n}\right)^{N-1} & \left(\frac{\alpha_l}{\alpha_l + \alpha_n}\right)^{N-2} & \cdots & \frac{\alpha_l}{\alpha_l + \alpha_n} & 1 \end{bmatrix} $$

Figure 4 A linear supply chain where each firm is a supplier to the firm towards its right.
The following proposition highlights the positive externalities of a firm’s investment on any other firm in the network.

**Proposition 2 (Positive externalities).** Assume \( \{3\} \) holds. The percentage change in disruption probabilities of firms with respect to a change in investment by firm \( i \) can be represented as a vector, 
\[
\delta^{(i)} = \frac{-1}{d_i} B_i,
\]
where \( \delta^{(i)}_j = \frac{1}{w_j} \frac{\partial w_j}{\partial y_i}, \) \( j = 1, 2, \ldots, N \) and \( B_i \) is the \( i \)th column of the \( B \) matrix.

The elements in \( B \) are all positive, thus there is a non-negative change in disruption probability of a firm when any other firm in its upstream network makes a positive investment. In other words, the investment game \( G(V, \{y_i\}, \{U_i\}, i \in V) \) is a game of strategic substitutes. This can be seen using Proposition 2 and by noting that \( \frac{\partial^2 U_i}{\partial y_i \partial y_j} \leq 0 \) for all \( i, j \).

Since the above result provides us with the network externalities for any investment vector, opportunities for mitigating inefficiencies exist even at the decentralized equilibrium. In particular, to overcome the inefficiency associated with unilateral investment, firms can decide to invest in other firms, provided the marginal benefit of such an investment exceeds the marginal cost. To illustrate, consider the example of three firms, 1, 2, and 3; with 1 supplying to 2, and 2 supplying to 3, and \( \alpha^0_i = 1 \forall i, \alpha_{12} = 1, \) and \( \alpha_{23} = 10. \) Using Proposition 2 it is possible to calculate \( \frac{\partial l_1 w_1}{\partial y_1} = -l_1 w_1, \frac{\partial l_2 w_2}{\partial y_1} = -l_2 w_2/2 \) and \( \frac{\partial l_3 w_3}{\partial y_1} = -(1/2)(10/11)l_3 w_3. \) Further, using Proposition 1 we know that at the decentralized equilibrium, \( l_1 w_1^* = d_1^* = 1; \) \( l_2 w_2^* = d_2^* = 2; \) and \( l_3 w_3^* = d_3^* = 11. \) Though there can be no further unilateral investment at the decentralized equilibrium, it is still possible for firms to consider collaborative approaches such as investing in other firms in order to mitigate their own risk. To see this, consider the marginal benefit for firm 3 by investing in firm 1 at equilibrium: 
\[
-\frac{\partial l_3 w_3}{\partial y_1} = (1/2)(10/11)11 = 5,
\]
is greater than the marginal cost (equals 1). Thus, the supply network creates a channel not only for propagating disruptions, but also for the positive externalities arising out of protective investment. For instance, in the related context of disruption risk from violation of corporate sustainability standards, some firms (e.g., Hewlett-Packard and Migros) actively manage the risk at higher-tier suppliers through supplier certification, factory audits, training programs, etc. (Huang et al. 2015).

Besides the opportunities for bilateral collaboration discussed above, it is equally easy to identify examples in which the collective marginal benefit to a group of firms by investing in an otherwise underinvesting network partner is greater than the marginal cost, thus creating opportunities for multilateral collaboration. Such collaborations are often observed in practice; e.g., in 2013, Wal-Mart Inc., Gap Retail Inc. and 17 other North American retailers set up a $42 million fund.
to improve safety conditions in the factories of Bangladesh. Earlier in the year, European retailers including H&M and Inditex pledged $60 million over five years for ensuring plant safety in Bangladesh. These measures were a follow-up to the twin disasters of a fire and building collapse in two factories that were producing for western retailers [Bloomberg 2013]. Successful execution of such initiatives additionally requires consideration of how the benefits would be divided amongst the coalition partners and which coalitions would be stable [Nagarajan and Sošic 2009, Sošic 2006]. Although the analysis of stable coalitions is beyond the scope of this paper, the result in Proposition 2 provides the basis to identify opportunities for such collaboration.

Building further on the theme of collaborative risk management, the most benefit can be extracted by adopting a centralized or welfare perspective: A powerful player (e.g., GM or Toyota in their respective supply chains) or a multilateral organization such as the Original Equipment Suppliers Association, could potentially play the role of the central coordinator. The following corollary identifies key players in the network, investing in whom can lead to the greatest decrease in the welfare (sum of individual firm payoffs) for the network, as a function of the network structure.

**Corollary 1.** The change in welfare in the network with respect to investment in firm $i$ is given by,

$$\frac{\partial \sum_{j \in V} U_j}{\partial y_i} = \frac{(l \circ w)^{\top} B_i}{d_i} - 1 = \frac{1}{d_i} \sum_{j \in V} l_j w_j B_{ji} - 1,$$

where $l \circ w$ is the element-wise product of vectors $l = [l_i]$ and $w = [w_i]$.

In the above result, the numerator, $(l \circ w)^{\top} B_i$, represents the expected losses incurred by the network due to a disruption to firm $i$. To see how, consider the expected loss of firm $j$ due to a disruption to firm $i$. This term can be written as $l_j \mathbb{P}(j \text{ is disrupted | } j \text{ is disrupted}) = l_j w_j B_{ji}$. Summing over all $j$s gives the expected losses due to $i$’s disruption as $(l \circ w)^{\top} B_i$. The corollary tells us that the relative impact of investment in different firms can be expressed as a ratio of the negative externality that a disruption to the firm causes to other firms in the network, to the in-degree of the firm, which is a measure of the responsiveness of a firm’s risk to protective investments. Thus, consistent with our intuition, we find that identifying the nodes which are important from a ‘global’ perspective (i.e., that of the central planner) calls for information about the entire network, as opposed to the decentralized investments in §4, which required limited local information.

It is interesting to note that our index for identifying the key player relates to similar indices arising in network studies in economics under different contexts. The term $(l \circ w)^{\top} B_i = (l \circ w)^{\top} (I - A)^{-1}$ can be construed to be a measure of centrality of node $i$ in the network. Indeed, the vector of weighted Bonacich centralities of nodes in a graph with (weighted) adjacency matrix $M \in \mathbb{R}^{n \times n}$,
and weight vector $l$ is given by $\rho(l, M) = l^\top (I - M)^{-1}$, provided $(I - M)^{-1}$ is well-defined and non-negative. Bonacich centrality is a common centrality measure that arises in many network settings to rank nodes based on their relative importance. In the words of Jackson (2010), this measure “presumes that the power or prestige of a node is simply a weighted sum of the walks that emanate from it.” In our context, the Bonacich centrality of a firm corresponds to the expected losses incurred by the network, when there is an external disruption to the firm. The key player metric in our model is also similar to the inter-centrality index for identifying key players proposed in Ballester et al. (2006); while both the indices are proportional to the Bonacich centralities, the scaling factor is different.

Using the results from this section, we next proceed to characterize the centralized solution to the investment problem, determining the inefficiency of the decentralized solution, and linking the network structure with inefficiency.

6. Inefficiency and Network Structure

6.1. Centralized decision-making

In this subsection, we characterize the inefficiency associated with decentralized decision making, by benchmarking it against a setting where the central planner makes optimal investment decisions on behalf of the firms. We consider the problem of a central planner whose objective is to minimize the aggregate expected losses of all firms. The payoff of the central planner will be $U_{cen}(y_i, i \in V) = \sum_i (-l_iw_i - y_i)$. For this problem and the remainder of this section, we continue to focus on the interior solution where the investments in nodes and links are nonzero at optimum; this is ensured by the constraints in $\text{CEN}$.

$$\max_{y_i; i \in V} U_{cen}(y_i, i \in V) = \sum_i (-l_iw_i - y_i) \text{ s.t. } (2) \text{ and } (3) \text{ hold for all } i. \quad (\text{CEN})$$

The following proposition characterizes the optimal solution of the central planner. Similar to the decentralized case, we derive expressions for expected losses, optimal investments and payoff for the central planner. Define matrix $\tilde{A}$ such that $\tilde{a}_{ij} = a_{ij}d_i^-$. Let $\gamma_i = d_i^- \log(d_i^-) - \alpha_i^0 \log(\alpha_i^0/\theta_i^0) - \sum_{j \in N_i} \alpha_{ji} \log(\alpha_{ji}/\theta_{ji}), \gamma = [\gamma_i]; \tilde{w}_i = \log(l_i/(d_i^- - d_i^+)), \tilde{w} = [\tilde{w}_i]$; and $D^- = \text{diag}(d_i^-)$.

**Proposition 3 (Central planner’s problem)**. The central planner’s problem $\text{CEN}$ is convex; and the unique optimal solution to the central planner’s problem is interior (the inequalities in (2) are satisfied), if and only if, for all $i$, $d_i^- > d_i^+$, and $l_i > \max\left(\frac{\alpha_i^0}{\theta_i^0}, \max_{j \in N_i} \frac{\alpha_{ji} l_j}{\theta_{ji} (d_i^- - d_j^+)}\right)$. Moreover, at the interior optimum:

- For every firm $i$, the weighted centrality $\rho_i(l \circ w^*, A)$ is equal to the weighted in-degree, $d_i^-$;
- The expected loss for firm $i$ is: $l_iw_i^* = d_i^- - d_i^+$;
• The vector of optimal investments made by the central planner, $y^*_\text{cen} = (D^* - \bar{A})\hat{w} + \gamma$.

Before we discuss the findings in Proposition 3, we investigate the conditions for the interior solution. The condition $d_i^- > d_i^+$ entails that the weighted in-degree must be greater than the weighted out-degree for all firms. This condition is satisfied for all nodes, if, for example, $\alpha_n$ (the inverse sensitivity for a node) is considerably higher than $\alpha_l$ (the inverse sensitivity for a link). Protecting against external disruptions is in general more difficult than warding off disruptions from suppliers and hence, we can expect link disruption probabilities to respond better to investment than node disruption probabilities. Under such circumstances, $\alpha_n > \alpha_l$ is a reasonable assumption.

The other condition, $l_i > \max\left(\frac{\alpha_i^0}{\bar{\alpha}_i^t}, \max_{j \in N_i} \frac{\alpha_{ji} l_j}{\bar{\theta}_{ji} (d_i^- - d_i^+)}\right)$, is similar to the conditions on losses for firms in decentralized settings, but for the fact that the minus term has been replaced with $d_i^- - d_i^+$. Since the central planner’s problem is a joint maximization over many variables, the conditions on losses for an interior solution are more stringent than in the decentralized case.

Turning to the findings in the proposition, we note that the weighted centrality is equal to the in-degree $d_i^-$ for every firm. The weighted centrality measure of firm $i$ is a measure of the expected losses to the network resulting from a disruption to $i$, i.e., $\rho(l \circ w, A) = (l \circ w)^\top (I - A)^{-1} = \sum_j l_j w_j B_{ji}$, where $B_{ji}$ is the sum of probabilities of the various paths of disruption from $i$ to $j$, given $j$ is disrupted. We can contrast this result with that from the decentralized solution, where the in-degree of a firm is equal only to its own expected losses. In the centralized setting, however, the optimal investment in firm $i$ also accounts for the (negative) externality that firm $i$ imposes on the entire network.

Further, we also find that the expected losses for a firm in this problem are equal to the difference of the weighted in- and out-degrees. Hence, the expected losses are lower for all firms in the centralized case, except for firms that are not others’ suppliers, e.g., firms that meet end-user demand. This observation brings a new perspective to the relationship between supply-chain resilience and network structure. While in the decentralized case, more connections are always bad for a firm, as they increase the firm’s in-degree; in the centralized case, an increase in connections does not matter as long as there is sufficient balance between the in- and out-degrees of a firm. However, we note that the reduction in expected losses in the centralized solution comes at the expense of increased investment.

Our next problem is to quantify absolute inefficiency, which we define to be the difference between the centralized payoff and the cumulative equilibrium payoffs in the decentralized solution.

### 6.2. Dependence of inefficiency on network structure

The following corollary to Propositions 1 and 3 characterizes the difference between the centralized and decentralized payoffs as a function of the network structure.
Corollary 2 (Comparison of decentralized and centralized payoffs). The differences in the aggregate investments and firm payoffs in the centralized and decentralized settings, when the solutions are interior, are as follows:

1. \( y_{cen} - y_{decen} = (D^T - \tilde{A})w^d \), where \( w^d_i = -\log\left(1 - \frac{d^+}{d_i}\right) \).

2. The inefficiency \( \Delta \) is given by: \( \Delta = U_{cen} - \sum_{i \in V} U_i = \sum_{i \in V} d_i^+ + \sum_{i \in V} (d_i^- - d_i^+) \log\left(1 - \frac{d^+}{d_i^-}\right) \).

Under- or over-investment in risk mitigation: Corollary 2 helps characterize the relationship between the position of firms in a network and the extent of under- or over-investments in risk mitigation. Restating the corollary, a firm under- (over-) invests in risk mitigation if \( w^d_i \) is greater (lesser) than \( \frac{1}{d_i^-} \sum_{j \in N_i} \alpha_{ji} w^d_j \). To understand this further, consider the ‘index’ \( w^d_i = -\log\left(1 - \frac{d^+}{d_i^-}\right) \).

This index is directly proportional to the out-degree of a firm, and since the out-degree of a firm represents the externalities it poses to the rest of the network that are not internalised by its decentralized investment decision, \( w^d_i \) could be construed to be a measure of the risk posed by a firm to the network. The term \( \frac{1}{d_i^-} \sum_{j \in N_i} \alpha_{ji} w^d_j \), on the other hand, represents the average risk faced by firm \( i \), weighted by the inverse sensitivities on the incoming links. Thus, intuitively, firms which face more risk than they pose to the network tend to over-invest in risk mitigation compared to the central optimum, and firms which face less risk from their upstream network than what they pose to their downstream network tend to under-invest in risk mitigation, and the indices \( w^d_i \), together with the inverse sensitivities help quantify the notions of under- and over-investments. Such a characterization can help firms identify weak nodes in the supply chain and improve resilience by providing incentives for such firms to invest in risk mitigation. From a risk management perspective, identification of the relationship between node position in a network and the level of underinvestment could aid the pricing of insurance products like the contingent business interruption (CBI) insurance, which serve to protect firms from disruptions arising elsewhere in the network.

A second utility of Corollary 2 is in establishing the relationship between inefficiency and network structure. Such a characterization is useful to identify the class of network topologies which stand to gain the most out of multilateral intervention. Comparison of networks, however, calls for appropriate normalization. Since the inefficiency in our model can be attributed to the inverse sensitivity terms (as seen in Corollary 2), we compare networks with the same total inverse sensitivity across the network. Under this normalization, any difference in inefficiency between networks can then be attributed solely to the variation in topology. In the following proposition, we establish the relationship between the inefficiencies of three standard topologies: a star-shaped assembly network, a star-shaped distribution network, and a linear network (Figure 4). We assume they have equal numbers of nodes (\(|V|\)) and equal numbers of edges (\(|E| = |V| - 1\)). We also set the node and link inverse sensitivities to be \( \alpha_n \) and \( \alpha_l \), respectively, for all nodes and links, thus normalizing the total inverse sensitivity to be \(|V|\alpha_n + (|V| - 1)\alpha_l\).
Proposition 4 (Star versus Linear). When the solutions are interior, the relationship between the inefficiencies of the assembly star, linear, and distribution star networks with identical values for sensitivities, and identical number of nodes and links, is given by: \( \Delta_{\text{assembly}} \geq \Delta_{\text{linear}} \geq \Delta_{\text{dist.}} \).

A rough intuition for the above result is as follows. In an assembly star, the hub (assembly) node has a large in-degree. In the decentralized setting, each of the spoke (supplier) nodes makes investment decisions while ignoring the externality it imposes on the assembly node. This leads to a high aggregate inefficiency in the network. By contrast, in a distribution star, the hub node acts only as a supplier to each spoke node, therefore it is not exposed to risk from their decisions. It is however true that for this topology, relative to the assembly star, the spoke nodes are exposed to risk due to underinvestment by the hub node, but that is still only one link that they have to worry about. Based on this intuition, we find that overall, the inefficiency associated with the assembly star is always higher than the inefficiency associated with the distribution star, with the linear network giving rise to a level of inefficiency that lies in between the two extremes.

Using the result in Proposition 4, one might conjecture that in more complicated network topologies, the greater the tendency to encounter assembly stars, the greater is the inefficiency; and greater the tendency to encounter distribution stars, the lower is the inefficiency; with more linear topologies resulting in intermediate levels of inefficiency. In the next section we introduce analysis using the methodology of random graphs that allows us to investigate this conjecture.

6.3. Sensitivity analysis of inefficiency

In this subsection, we study the dependence of inefficiency on different network characteristics. While our analysis thus far has provided an exact characterization of inefficiency for a given network structure, a complete characterization of the relationship calls for an understanding of the variation of inefficiency as a function of common graph properties, such as connectivity, size, parameters of degree distributions, etc. A useful tool for this purpose is the theory of random graphs, which provides us with guidelines on generating random instances of graphs possessing desired properties. The average inefficiency over such instances can be used to glean insights on the variation of inefficiency with the graph property of interest. While we have conducted extensive numerical analyses to generate insight, for brevity, we report only one illustrative set of results for each graph property that we investigate.

Our first observation concerns the variation of inefficiency with inverse sensitivity parameters, and can be deduced analytically from Corollary 2. The intuition involved is simple. As \( \alpha_n \) increases, the problem of investing in nodes to reduce disruption cascades becomes more difficult, compared to that of investing in links to prevent external disruptions. Since both the decentralized and
centralized problems are equivalent in the absence of cascades, the inefficiency decreases with increases $\alpha_n$. A similar argument can explain the variation with $\alpha_l$.

**Observation 1.** Average inefficiency decreases with increase in $\alpha_n$ (for fixed $\alpha_l$) and increases with $\alpha_l$ (for fixed $\alpha_n$).

In keeping with this observation, we find that the average inefficiency increases with network size ($|V|$) and connectivity, which we define to be the ratio, $|E|/|V|$. To illustrate the change in inefficiency with respect to changing size and connectivity, we appeal to the theory of random graphs. We use the Erdős-Rényi model to generate random graphs with a specific number of nodes and edges, wherein we pick a graph uniformly at random from the set of ‘candidate’ graphs. As before, in order to zero in on the effects of network structure, we fix the inverse sensitivity parameters to be same for all nodes ($\alpha_n = 10$) and links ($\alpha_l = 1$). We generate instances of connected and directed random graphs with $|V| = 30$ and plot average inefficiency of these instances as a function of connectivity. The result of our simulations is shown in Figure 5(a). In Figure 5(b) we plot the average inefficiency as a function of $|V|$, while keeping connectivity fixed at 2.5.

In our second task, we consider a parametrized family of graphs and study the dependence of inefficiency on the parameters. Recent empirical studies on supply chain networks (for example, Wu and Birge (2014)) observe a power law trend in the in- and out-degree distributions of nodes in supply chains. The power law trend is commonly observed in the literature of networks when there is preferential attachment; that is, if we ‘grow’ supply chains from scratch, a new supplier is more likely to supply to a buyer with many existing suppliers, and a new buyer is likely to buy from a supplier already supplying to many firms. In line with this observation, we assume power law distributions for the (weighted) in- and out-degrees and calculate the dependence of the average inefficiency on the power-law exponents. For the power law, we assume a generalized Pareto distribution on all positive reals. Ideally, we require a discrete distribution for the degrees over integers (which for the power law is given by the zeta distribution), but we opt for a continuous distribution to keep the analysis simple. The generalized Pareto distribution is defined ($\rho_0^x$ and $\xi_x$, which are, respectively, the probability of having a firm having an in- or out-degree of zero and the power-law exponent ($\xi_x \geq 0$)). Moreover, the threshold $t_x$ is $\alpha_n$ and zero for in- and out-degree distributions, respectively.

$$f(d_x) = \rho_0^x \left(1 + \frac{\rho_0^x(d_x - t_x)}{\xi_x}\right)^{-1-\xi_x}, \quad d_x \geq t_x \text{ and } x = \{\text{in, out}\}$$

Average inefficiency can then be computed by evaluating a double integral, i.e., integrating $\Delta$ over the in- and out-degree distributions. From Corollary 2, each node’s contribution to $\Delta$ is

---

9 Random graphs were generated in Mathematica using the *RandomGraph* command.
\[ d_i^+ + (d_i^- - d_i^+) \log \left( 1 - \frac{d_i^+}{\alpha_i} \right). \]

Using the distributions, we numerically calculate the average per-node inefficiency as a function of the in- and out-degree exponents and plot the relations in Figure 6. To make sense of the average inefficiency, we vary the in- and out-degree exponents and plot the change in average inefficiency, keeping other model parameters fixed. We note that there can potentially be parameter values for which no corresponding graph exists (Chen and Olvera-Cravioto 2013) – constructing random graphs, given in- and out-degree distributions, is known to be a challenging graph-theoretic problem – and hence, our approach to calculating the average inefficiency can be viewed as one involving numerical relaxations of the graph structure.

**Observation 2.** Average inefficiency increases with an increasing in-degree exponent and decreases with an increasing out-degree exponent.

To interpret this observation, we need to draw the connection between the in- and out-degree exponents and network structure. A high in-degree exponent means that the probability distribution of the in-degree is more heavily concentrated on smaller numbers of nodes, i.e., there are many nodes with very low in-degree, and the number of nodes with high in-degree is very small. An extreme example of this will be the assembly star topology, where the spoke nodes have an in-degree of \( \alpha_n \) and the hub node is the only one with an in-degree of \( \alpha_n + (N-1)\alpha_l \). On the other hand, a low in-degree exponent will ensure that the in-degrees are more or less the same for all firms; the linear network being a case in point. Similarly, an example for high out-degree exponent is that of a star-shaped distribution network (in contrast to the star-shaped assembly network in the case of the high in-degree exponent), where a single hub firm supplies to many spoke firms.

In the previous section, intuition from Proposition 4 had led us to conjecture that inefficiency in complex networks would depend on whether the network would be more likely to contain assembly stars, distribution stars, etc. The result in Observation 2 is broadly consistent with this conjecture. Figure 7 provides the overall summary of this section.

**Figure 5** Variation of average inefficiency \( \Delta \) (and the 95% confidence interval) as a function of graph connectivity (#edges/#nodes) and as a function of graph size (#nodes). The parameter values are \( \alpha_n = 10, \alpha_l = 1 \) and the averages were computed over 1000 instances.
2.555
2.560
2.565
2.570
2.575
ξ_{in}

Figure 6 Variation of average inefficiency \( \Delta \) with in-degree and out-degree exponents, \( \xi_{in} \) and \( \xi_{out} \). The parameter values are \( \alpha_n = 10, \alpha_l = 1, \rho_0 \rightarrow \rho_0 = 0.4 \).

\[
\begin{align*}
\text{Inefficiency } \Delta & = \frac{1}{3} \\
\text{Increasing } \xi_{out} & \quad \text{Increasing } \xi_{in}
\end{align*}
\]

Figure 7 Inefficiency and graph structure. The average inefficiency increases with \( \xi_{in} \) and decreases with \( \xi_{out} \).

7. Extensions

In this section, we relax some of the assumptions of the model, and describe the corresponding implications.

7.1. Relaxation of assumption on interior solution

Throughout our analysis, we have focused on the interior solution, i.e., the investment that firm \( i \) makes is large enough to ensure that all link and node investments are strictly positive (at least 1 pence). We now examine the effect of relaxing this assumption. Firm \( i \) invests an amount which maximizes its payoff, and this investment might result in zero investment along one or more links or on the node. In the following proposition we examine the impact of zero investment at optimum on a particular link.

**Proposition 5.** Consider a supply chain where \( i \) is a supplier to \( j \) and \( j \) is a supplier to \( k \). If the optimal investment \( y_{ij}^* = 0 \) along link \( i \rightarrow j \), then at the decentralized equilibrium, the disruption probability \( w_{jk}^* \) depends on \( w_{ij}^* \).

Thus, we identify the following change in intuition from the case of interior solution: along links where no investment has been made, the second-tier firms play the role of the first-tier firms. In
terms of information requirements for optimal decision making, for the example provided here, firm $i$ would need to know the weighted in-degree of all of the suppliers to firms $j$ and $k$, in addition to the weighted in-degrees and losses of its other suppliers. Besides, the weighted in-degree of firm $i$ excludes the inverse sensitivities on links $j \rightarrow i$ and $k \rightarrow i$. Therefore we conclude that our main insights continue to hold when the assumption of an interior solution is relaxed.

7.2. Budget constraints

In our analysis, we have not placed any restrictions on firms’ availability of funds; firms are able to invest as much in risk mitigation as determined by their first-order conditions. We relax this restriction here and analyze the changes in the decentralized equilibrium.

Suppose a firm is budget-constrained and is unable to make the entire investment as governed by the first-order conditions. Mathematically, the lower bound dictated by (2) holds, but in addition, there is an upper bound $y_i \leq b_i$, that caps the investment made by firm $i$. We can see that the expected losses in equilibrium in this scenario depend on both the equilibrium investment in the absence of budget, $y^*_i$, and the budget $b_i$. We consider two cases: firstly if $b_i \geq y^*_i$, the budget will not be binding, the firm will invest $y^*_i$ as before, and our insights and results carry through. On the contrary, suppose $b_i < y^*_i$, then let $k = y^*_i/b_i$. In this case, the budget $b_i$ is the optimal investment for firm $i$, as the objective function monotonically increases until the equilibrium point. At optimum, the expected loss $l_i w^*_i = kd_i^-$. That is, the expected loss, and hence payoff, still depends only on three local factors of firm $i$ - budget, weighted in-degree, and the equilibrium investment in the absence of a budget. This logic can be extended to the case where multiple firms in the network are budget constrained. Starting with firms with no suppliers, the logic described above is adequate. Thereafter, the argument can be recursively applied to firms that are further down the network to show that the expected losses and total payoff depend only on properties of the extended local neighborhood, which now include information about the budget constraints on nodes in this neighborhood. Thus, we conclude that the presence of budgetary constraints does not alter our core insight that just local network information is relevant for the firm’s investment decision.

7.3. Substitutes

While the main focus of the model is on disruptions from complementary products, in general, a firm will dual-source some components. Studying disruption cascades in the presence of both complementary and substitute suppliers requires a significantly more sophisticated model. A key challenge in the presence of substitutes is that based on the fraction of inputs sourced from different suppliers, production capacity of firms may be partially disrupted, and the 0/1 disruptions described in our model are inadequate in this aspect.
We investigate two ways by which our model can be used to incorporate the presence of substitute suppliers. Firstly, the essence of substitutes can be captured by assigning low values to the inverse sensitivities of links along which alternate suppliers are available. Alternatively, if we assume that the events of cascading disruptions from different suppliers to $i$ are independent, then the disruption probability $w_i$ when all suppliers $j \in N_i$ provide substitute components can be seen to be:

$$w_i(y_i, w_j, j \in N_i) = \min_{y_i^0 + \sum_{j \in N_i} y_{ij} = y_i} \theta_i^0 e^{-y_i^0/\alpha_i^0} + \prod_{j \in N_i} \theta_{ji} e^{-y_{ji}/\alpha_{ji}}$$

In the presence of substitutes, firm $i$ is disrupted only when all its suppliers are disrupted; the independence assumption calls for the replacement of the summation over suppliers $j \in N_i$ in our original formulation with a product. While it would be consistent with good management practice if firms identify substitute suppliers whose disruptions are independent, this modeling approach breaks down when the suppliers’ disruptions are not independent (say, due to a common second-tier supplier).

8. Conclusion
In this paper, we considered the impact of network structure on optimal decision making to minimize the threat of disruptions in supply chain networks. We studied the optimal solution under both decentralized and centralized decision making settings, and we found that for making optimal decisions, a firm requires information only about its extended local neighborhood, i.e., up to its tier-2 suppliers. By characterizing the positive externalities of one firm’s investment on another firm’s payoff, we were able to develop a scheme to identify opportunities for collaborative risk management. We were also able to analytically characterize supply-chain resilience, as well as the absolute inefficiency that arises due to decentralized investment, and to relate these metrics to network topology.

We believe there are a few potentially rewarding directions to take this work forward. One such direction is to study coordination mechanisms between firms in the network in order to ensure efficient investments. Such mechanisms may involve the study of stable coalitions of firms in a network, and may generate insights about the relationship about the network structure and the stability of coalitions. The question of how coordination might be achieved with limited network visibility (e.g., when firms only know about their adjacent tiers) also remains to be explored. Another promising avenue for research pertains to the dynamic treatment of disruptions. Our model incorporates a one-shot treatment, but the real mechanics of disruptions are dynamic, as firms could recover and be disrupted by a second supplier. Capturing the entire dynamics of disruptions evolving over time presents considerable challenge; however, such a model could generate useful insights on optimal operational decisions to be made during the course of disruptions.
References


Stochastic Systems 3(1) 147–186.

de Martí, Joan, Yves Zenou. 2013. Network games with incomplete information. Tech. rep. URL http:

DeCroix, Gregory A. 2013. Inventory management for an assembly system subject to supply disruptions.
Management Science 59(9) 2079–2092.

236–249.


Economic Review 104(10) 3115–53.

9ac4d7e2-595f-11e0-bc39-00144feab49a.html.

assets/pdf/P10168.pdf


games. The Review of Economic Studies 77(1) 218–244.

Press.


long-run stock price performance and equity risk of the firm. Production and Operations management
14(1) 35–52.

paper.


Appendix A: Proofs of results

Proof of Lemma 1

We shall solve the optimisation problem by assuming the inequalities are not strict, and later get conditions for interior solution. Clearly the objective function is convex in the investments \((y_i^0, y_{ji}, j \in N_i)\) and the constraint set is compact. The first-order KKT conditions are necessary and sufficient for optimality. We now characterize the unique global minimum by solving the first-order conditions. The Lagrangian of the problem can then be written as:

\[ L(\lambda, y_i^0, y_{ji}, j \in N_i) = \theta_i^0 e^{-y_i^0/\alpha_i^0} + \sum_{j \in N_i} w_j \theta_{ji} e^{-y_{ji}/\alpha_{ji}} + \lambda (y_i^0 + \sum_{j \in N_i} y_{ji} - y_i), \]

where \(\lambda \in \mathbb{R}\) is the Lagrange multiplier. The first order conditions for interior solution is:

\[ \lambda = (\theta_i^0/\alpha_i^0) e^{-y_i^0/\alpha_i^0} = (w_j \theta_{ji}/\alpha_{ji}) e^{-y_{ji}/\alpha_{ji}}. \]

For an optimal interior solution \((y_i^0 > 0 \text{ and } y_{ji} > 0, \forall j \in N_i)\), we need: \(\theta_i^0/\alpha_i^0 > \lambda\) and \(w_j \theta_{ji}/\alpha_{ji} > \lambda\). Under this condition, \(\lambda\) can be evaluated from the equality constraint as:

\[ y_i = -\alpha_i^0 \log(\lambda \alpha_i^0/\theta_i^0) - \sum_{j \in N_i} \alpha_{ji} \log(w_j \theta_{ji}) \frac{\lambda \alpha_{ji}}{w_j \theta_{ji}} \]

\[ \log(\lambda) = \frac{-y_i + \alpha_i^0 \log(\lambda \alpha_i^0/\theta_i^0) + \sum_{j \in N_i} \alpha_{ji} \log(\alpha_{ji}/(w_j \theta_{ji}))}{\alpha_i^0 + \sum_{j \in N_i} \alpha_{ji}} \]

Subsequently, the optimal investments \(y_i^0\) and \(y_{ji}, j \in N_i\) can then be computed as follows.

\[ y_i^0 = \alpha_i^0 \left( \log \left( \frac{\theta_i^0}{\alpha_i^0} \right) + \frac{y_i + \alpha_i^0 \log(\lambda \alpha_i^0/\theta_i^0) + \sum_{j \in N_i} \alpha_{ji} \log(\alpha_{ji}/(w_j \theta_{ji}))}{\alpha_i^0 + \sum_{j \in N_i} \alpha_{ji}} \right) \]

\[ y_{ji} = \alpha_{ji} \left( \log \left( \frac{\theta_j \theta_{ji}}{\alpha_{ji}} \right) + \frac{y_i + \alpha_i^0 \log(\lambda \alpha_i^0/\theta_i^0) + \sum_{j \in N_i} \alpha_{ji} \log(\alpha_{ji}/(w_j \theta_{ji}))}{\alpha_i^0 + \sum_{j \in N_i} \alpha_{ji}} \right) \]

Under the conditions for interior solution, the disruption probability \(w_i\) reduces to:

\[ w_i = (\alpha_i^0 + \sum_{j \in N_i} \alpha_{ji}) \exp \left( \frac{y_i - \alpha_i^0 \log(\lambda \alpha_i^0/\theta_i^0) - \sum_{j \in N_i} \alpha_{ji} \log(\alpha_{ji}/(w_j \theta_{ji}))}{\alpha_i^0 + \sum_{j \in N_i} \alpha_{ji}} \right) \]

\[ w_i = d_i^- \exp \left( \frac{-y_i}{d_i^-} \right) \frac{\theta_j \theta_{ji}}{\alpha_{ji}} \frac{\alpha_{ji}}{\theta_i} \frac{\alpha_i^0}{\alpha_i^0 \exp(-by_i)} \]

Proof of Proposition 1

For an investment of \(y_i\), the payoff of firm \(i\) is given by \(U_i = -l_i w_i - y_i\). The optimal investment \(y_i\) to be made by the firm is derived from first-order conditions as \(-l_i \frac{\partial w_i}{\partial y_i} = 1\). Since the positive externalities of a firm’s investment does not impact its suppliers, \(\frac{\partial w_i}{\partial y_i} = 0\) for all \(j \in N_i\). From the result in Lemma 1, we note that the form of the payoff function is \(U_i = -a \exp(-by_i) - y_i\) with \(a, b > 0\). For such a payoff function, the optimal investment \(y_i^* = \log(ab)\) if \(ab > 1\) and zero otherwise. The optimal investment \(y_i^*\) can be written as follows.

\[ y_i^* = \begin{cases} d_i^- \log(l_i) + \sum_{j \in N_i} \alpha_{ji} \log \left( \frac{w_j \theta_{ji}}{\alpha_{ji}} \right) + \alpha_i^0 \log \left( \frac{\theta_i^0}{\alpha_i^0} \right), & \text{if } l_i > \Pi_{j \in N_i} \left( \frac{\alpha_{ji}}{w_j \theta_{ji}} \right)^{\alpha_{ji}/d_i^-} \frac{\alpha_i^0}{\alpha_i^0 \exp(-by_i)}; \\ 0, & \text{otherwise} \end{cases} \]
Substituting this for \( y_i \) in the expression for \( w_i \) in Equation 3, we get the equilibrium disruption probabilities.

\[
w_i^*(y_i^*) = \frac{d_i}{l_i}, \quad \text{if } l_i > \Pi_{j \in N_i} \left( \frac{\alpha_{ji}}{w_j^0 \theta_{ji}} \right)^{\alpha_{ji}/d_i} \left( \frac{\alpha_i^0}{\theta_i^0} \right)^{\alpha_i^0/d_i}
\]

We end the proof by commenting on the conditions for the interior solution, \( y_i^0 > 0 \) and \( y_j^* > 0, \ j \in N_i \), (which are equivalent to \( \theta_i^0 / \alpha_i^0 > \lambda \) and \( w_j^0 \theta_{ji} / (\alpha_{ji} > \lambda) \), when firm \( i \) makes the optimal investment \( y_i^* \). If we substitute the expression for optimal \( y_i^* \), the conditions reduce to \( l_i > \frac{\alpha_i}{w_i^0 \theta_{ji}} \) and \( l_i > \frac{\alpha_{ji}}{w_j^0 \theta_{ji}} \), \( \forall j \in N_i \) respectively for \( y_i^0 > 0 \) and \( y_j^* > 0 \). Since the condition \( l_i > \Pi_{j \in N_i} \left( \frac{\alpha_{ji}}{w_j^0 \theta_{ji}} \right)^{\alpha_{ji}/d_i} \left( \frac{\alpha_i^0}{\theta_i^0} \right)^{\alpha_i^0/d_i} \) is subsumed by these conditions put together, strict positivity of all link and node investments is guaranteed by \( l_i > \max \left( \frac{\alpha_i}{w_i^0 \theta_{ji}}, \frac{\alpha_{ji}}{w_j^0 \theta_{ji}} \right) \). Moreover, under such a condition, the link disruption probabilities at equilibrium are \( p_{ji} = \frac{\alpha_{ji}}{\alpha_i} \) and the node disruption probability is \( p_i^0 = \frac{\alpha_i^0}{l_i} \).

**Proof of Lemma 2**

This proof uses Bayes’ theorem and the result from Proposition 1. By Bayes’ rule, the conditional probability that firm \( j \) disrupted firm \( i \) given firm \( i \) is disrupted is given by

\[
p_{ji} w_j = \frac{p_{ji} w_j}{\sum_{k \in N_i} p_k w_k}.
\]

From the proof of Proposition 1, we have,

\[
p_{ji} w_j = \frac{p_{ji} w_j}{p_i^0 + \sum_{k \in N_i} p_k w_k} \\
= \frac{\alpha_{ji} \exp \left( \frac{y_i - \alpha_i^0 \log \left( \frac{\alpha_i}{\theta_i} \right) - \sum_{j \in N_i} \alpha_{ji} \log \left( \frac{w_j^0 \theta_{ji}}{\alpha_{ji}} \right)}{d_i} \right)}{\alpha_i^0 + \sum_{k \in N_i} \alpha_k \exp \left( \frac{y_i - \alpha_i^0 \log \left( \frac{\alpha_i}{\theta_i} \right) - \sum_{j \in N_i} \alpha_{ji} \log \left( \frac{w_j^0 \theta_{ji}}{\alpha_{ji}} \right)}{d_i} \right)} \\
= \frac{\alpha_{ji}}{d_i}
\]

**Proof of Proposition 2**

To denote the change in probabilities with respect to \( y_i \), let \( \delta^{(i)} \) be a vector such that

\[
\delta_j^{(i)} = \frac{\partial \log (w_i)}{\partial y_i}
\]

Differentiating the expression for \( w_i \) with respect to \( y_i \) and \( y_j \), we have,

\[
\frac{\partial w_i}{\partial y_i} = \frac{-1}{\alpha_i^0 + \sum_j \alpha_{ji}} \\
\frac{\partial w_j}{\partial y_i} = \sum_{k \in N_j} \frac{\alpha_k}{\alpha_{ji}^0 + \sum_{l \in N_j} \alpha_l} \frac{\partial w_k}{\partial y_i} \\
\delta_j^{(i)} = \sum_k a_{jk} \delta_k^{(i)}
\]

The first equation holds because positive externalities move downstream only, i.e., a firm’s investments will not benefit its suppliers. The above equations can be written in matrix form as:

\[
\delta^{(i)} = b + A \delta^{(i)}
\]

where \( b \) is a vector whose all components are zero except for \( b_i = \frac{1}{\alpha_i^0 + \sum_{j \in N_i} \alpha_j} \) and \( A = [a_{ij}] \), with

\[
a_{ij} = \frac{\alpha_{ji}}{\alpha_i^0 + \sum_{l \in N_i} \alpha_l}, \quad \text{if } j \in N_i, \text{ and zero otherwise, and } a_i = 0.
\]

To prove the existence of the inverse, note that since \( \alpha_i^0 > 0 \) \( \forall i \), \( I - A \) is a strictly diagonally dominant matrix. By the Levy-Desplanques theorem, \( I - A \) is non-singular, thus guaranteeing the existence of a unique inverse.
Proof of Corollary 1: We take the partial derivative of $\sum_{j \in V} U_j$, and use the result from Proposition 2

$$\sum_{j \in V} U_j = \sum_{j \in V} -l_j w_j - y_j$$

$$\frac{\partial \sum_{j \in V} U_j}{\partial y_i} = \sum_{j \in V} -l_j \frac{\partial w_j}{\partial y_i} - 1$$

$$= \sum_{j \in V} -l_j w_j / w_j - 1$$

Another application of Proposition 2 yields $\partial (\sum_{j \in V} U_j)/\partial y_i = \sum_{j \in V} (l_j w_j / d_i^-) B_{ji} - 1 = (1 / w_i) B_{ij} - 1$.

Proof of Proposition 3: We first prove that the central planner’s problem [CEN] is convex. To this end, we show that each of the $w_j$s are convex functions of $y$. To see this, take logarithm of $w_i$ in Equation 3 and find that $\log(w_i)$ can be expressed as a linear function of the $y_i$s, i.e., $\log(w_i) = \sum_{j \in N_i} \alpha_{ji} / d_i^- \log(w_j) + b_i - y_i / d_i^-$, for some constant $b_i$. Writing this in the form of a matrix, $\log(w) = A \log(w) + b - \tilde{y}$, where $\tilde{y}_i = y_i / d_i^-$. Since $I - A$ is invertible, $\log(w_i)$ can be expressed as an affine function of the $y_i$s, and consequently $w_i = a_i \exp(\sum b_{ji} y_j)$, for some $a_i$s and $b_{ji}$s. The convexity of these functions ensures the convexity of the objective function. Since $\log(w)$s are affine functions of $y$, the inequalities in Equation 2 define a convex set. Hence the central planner’s problem is a convex optimisation problem.

The central planner’s payoff function is

$$U_{cen} = \sum_{k \in V} (-l_k w_k - y_k)$$

with constraints in Equation 2. The first-order KKT conditions are

$$\sum_k l_k \frac{\partial w_k}{\partial y_i} + 1 - \lambda_i = 0,$$

for $i = 1, 2, \ldots, N$, where $\lambda_i$ is the Lagrange multiplier corresponding to the constraint $y_i \geq 0$. From the proof of Proposition 2, we have that

$$\frac{\partial w_k / w_k}{\partial y_i} = -B_{ki} / d_i^- = -(I - A)_{ki}^{-1} / d_i^-.$$

Plugging this in the FOC, we have

$$\sum_k l_k w_k (I - A)^{-1}_{ki} = d_i^- (1 - \lambda_i), \forall i.$$

Moreover for an interior solution with $y > 0$, $(I \circ w)^T B = (I \circ w)^T (I - A)^{-1} = d^{-T}$. Simplifying this gets us

$$l_i w_i = [(I - A)^T d^-]_i = a_i^0 + \sum_{j \in N_i} \alpha_{ji} - \alpha_i \sum_{j \in V} (1 - \lambda_j) = c_i^- - d_i^+,$$

for all $i$. In other words, when the weighted out-degree of all firms is less than the weighted in-degree of all firms, the expected loss for all firms is equal to the difference between the two in the central planner’s optimal solution.

Since [CEN] is convex, the first-order conditions are necessary and sufficient. A solution to the first-order conditions exists if and only if $l_i w_i^* = c_i^- - d_i^+ > 0$. When node inverse sensitivities are lower than their link counterparts, firms may invest only in nodes, thus rendering an interior solution suboptimal. Substituting
the optimal $w^*_i$ in the Equation 2 gives us the condition on the loss terms to guarantee an interior solution. For part (ii), we use our definition of centrality $\rho_i = \sum_k l_k w_k B_{ki}$.

To find the optimal investments in part (iii), note:

$$y^*_{cen} = d^*-d^+_i \log(d^+_i/w_i) + \sum_{j \in N_i} \alpha_{ji} \log \left(\frac{w_j \theta_{ji}}{\alpha_{ji}}\right) + \alpha^0 \log(\theta^0_i/\alpha^0_i)$$

$$= \gamma_i - d^*_i \log(w_i) + \sum_{j \in N_i} \alpha_{ji} \log(w_j)$$

Substituting $w^*_i = \frac{d^*_i - d^+_i}{\gamma_i}$ and simplifying, we find $y_{cen} = (D^* - \hat{A})\hat{w} + \gamma$.

**Proof of Proposition 4** The inefficiencies of star and linear topologies are as follows.

$$\Delta_{\text{assembly}} = |E|\alpha_i + |E|(\alpha_n - \alpha_i) \log \left(1 - \frac{\alpha_i}{\alpha_n}\right)$$

$$\Delta_{\text{line}} = |E|\alpha_i + (\alpha_n - \alpha_i) \log \left(1 - \frac{\alpha_i}{\alpha_n}\right) + (|E| - 1)\alpha_n \log \left(1 - \frac{\alpha_i}{\alpha_n + \alpha_i}\right)$$

To prove the result, we need to show: $(\alpha_n - \alpha_i) \log \left(1 - \frac{\alpha_i}{\alpha_n}\right) \geq \alpha_n \log \left(1 - \frac{\alpha_i}{\alpha_n + \alpha_i}\right)$. Since $\alpha_n > \alpha_i$ by the requirement of interior solution in the central planner’s problem, divide both sides by $\alpha_n$ and set $z = \alpha_i/\alpha_n$.

Now we are left with comparing two functions $f(z) = -\log(1+z)$ and $g(z) = (1-z)\log(1-z)$. It is easy to see that $f(z) \leq g(z)$ for $0 \leq z \leq 1$. The argument can be extended to the star distribution network by observing that $\Delta_{\text{star}} = |E|\alpha_i + (\alpha_n + |E|\alpha_i) \log \left(1 - \frac{|E|\alpha_i}{\alpha_n + \alpha_i}\right)$.

**Proof of Proposition 5** First, we extend Lemma 1 to compute the optimal $w_i$ when the solution is not in the interior, that is, if one or more of $y^0_i$ or $y_{ji}$ are zero at optimum. The KKT conditions in the proof of the lemma tell us that in an interior solution, the values $\theta^0_i/\alpha^0_i \exp(-y^0_i/\alpha^0_i)$, and $w_i \theta_{ji}/\alpha_{ji} \exp(-y_{ji}/\alpha_{ji})$ are equal. Suppose $y_{ji} = 0$ at optimum, then $\lambda = w_i \theta_{ji}/\alpha_{ji} + \mu_{ji}$, where $\mu_{ji} > 0$. Thus, based on the factors $\theta^0_i/\alpha^0_i$ and $w_i \theta_{ji}/\alpha_{ji}$ and the total investment $y_i$, we propose an iterative approach to solve the KKT conditions. This procedure is detailed in Algorithm 1. Let us sort the quantities $\theta^0_i/\alpha^0_i$, $w_i \theta_{ji}/\alpha_{ji}$, $j \in N_i$ as an ascending sequence of numbers $a_{(j)}$, $j = 0, 1, \ldots, |N_i|$. Let $\alpha_{(j)}$ be the $\alpha$-term in the denominator of the number $a_{(j)}$, e.g., if $a_{(y)} = w_k \theta_{ki}/\alpha_k$ where $k \in N_i$, then $\alpha_{(y)} = \alpha_k$.

$l^*$ is the number of avenues of investment of firm $i$ (out of the node and $N_i$ links) that have zero investment at optimum. If $l^* = 0$, then the optimal solution is in the interior and all links and the node receive positive investment at optimum. However, as $y_i$ decreases, $l^*$ increases from 0 and the interior solution can no longer be satisfied. For example, if $l^* = 1$, and $a_{(0)} = w_k \theta_{ki}/\alpha_k$, then there is no investment on the link $k \rightarrow i$ in the optimal solution (i.e., $y^*_{ki} = 0$), but non-zero investment in the node and the rest of the links. The continuity and piecewise-convexity properties can be inferred from the functional form of $w^*_i$, which can be computed from the expression for $\log(\lambda)$. The term $w_i = \sum_{k=0}^{l^*-1} a_{(k)} \alpha_{(k)} + \kappa_{l^*} \exp(-y_i/\kappa_{l^*}) \Pi_{k=l^*}^{N_i} a_{(k)}^{\alpha_{(k)}/\kappa_{l^*}}$ can be used in place of (3) in §5 to compute the externalities. To find the optimal investment in Proposition 2 the first-order condition is $l_i w_i'(y_i') = -1$, and let us assume at the optimum, $l^*$ avenues have zero investment. The new solution is:

$$1/l_i = \exp(-y_i'/\kappa_{l^*}) \Pi_{k=l^*}^{N_i} a_{(k)}^{\alpha_{(k)}/\kappa_{l^*}}$$
Algorithm 1 Finding optimal \( y^*_i, y_{ji}, j \in N_i \), given \( y_i \)

1: \textbf{procedure}
2: \hspace{1em} \textbf{for} \( l = 0 \) to \(|N_i| - 1\) \textbf{ do}
3: \hspace{2em} Compute \( \log(\lambda) = -\frac{y_i + \sum_{k=l}^{\lvert N_i \rvert - 1} \alpha(k) \log(a(k))}{\sum_{k=l}^{\lvert N_i \rvert - 1} \alpha(k)} \)
4: \hspace{2em} If \( \lambda < a(l) \), break;
5: \hspace{1em} \textbf{end for}
6: \( l^* = l \);
7: \( y^*_{(m)} = 0 \) for \( m = 0, 1, \ldots, l^* - 1 \);
8: \( y^*_{(m)} = \alpha(m) \log(a(m)/\lambda) \) for \( m = l^*, l^* + 1, \ldots, \lvert N_i \rvert \);
9: Define \( \kappa_{l^*} = \sum_{k=l^*}^{\lvert N_i \rvert} \alpha(k) \);
10: \( w_i = \sum_{k=0}^{l^*-1} a(k) \alpha(k) + \kappa_{l^*} \exp(-y_i/\kappa_{l^*}) \prod_{k=l^*}^{\lvert N_i \rvert} \frac{\alpha(k)/\kappa_{l^*}}{a(k)} ; \)
11: \textbf{end procedure}

\[
\begin{align*}
y^*_i &= \kappa_{l^*} \left( \log(l_i) + \sum_{k=l^*}^{\lvert N_i \rvert} \frac{\alpha(k)}{\kappa_{l^*}} \log(a(k)) \right) \\
\kappa_{l^*} &= l_i \left( w^*_i - \sum_{k=0}^{l^*-1} a(k) \alpha(k) \right)
\end{align*}
\]

At the optimum \( y^*_i \), we observe that the in-degree \( \kappa_{l^*} \) is effectively changed to include the inverse sensitivities of only those links where a non-zero investment is made. Suppose \( k \rightarrow i \in E \) and \( i \rightarrow j \in E \). From the last equation, we observe that, in case, no investment has been made on link \( k \rightarrow i \), i.e., \( y^*_{ki} = 0 \), at optimum, \( w^*_j \) depends on \( w^*_i \), and subsequently \( w^*_k \). This is a departure from the interior solution case, where \( w^*_i = d^-_i/l_i \), and the disruption probability \( w^*_j \) was independent of \( w^*_k \), \( k \in N_i \).
Appendix B: Allowing for co-dependence between investments

In this Appendix, we shall relax the assumption that investment on a particular link serves only to reduce the conditional probability of disruption on the same link. To allow for such a co-dependence between investments along different links at a firm, we assume the following functional dependence of $w_i$ on the investments holds.

$$w_i = \min_{y_i^0 + \sum_{j \in N_i} y_{ji} = y_i} \theta_i^0 \exp \left( - \left( \frac{y_i^0}{\alpha_i^0} - \sum_{j \in N_i} \beta_{0j} y_{ji} \right) \right) + \sum_{j \in N_i} w_j \theta_{ji} \exp \left( - \left( \frac{y_{ji}}{\alpha_{ji}} - \beta_{0j} y_i^0 - \sum_{k \in N_i \setminus j} \beta_{jk} y_{ki} \right) \right)$$

The new terms $\beta$ serve to capture the extent of influence the investment on a link has on the conditional probability of disruption on another link. Specifically, $\beta_{jk}$ captures the effect of investment $y_{ki}$ on the conditional probability of disruption along link $j \rightarrow i$, i.e., $w_j \theta_{ji}$. We solve for the optimal investments, in the same way, as we did in the proof of Lemma 1. The Lagrangian and the FOC are as follows.

$$L = \theta_i^0 \exp \left( - \left( \frac{y_i^0}{\alpha_i^0} - \sum_{j \in N_i} \beta_{0j} y_{ji} \right) \right) + \sum_{j \in N_i} w_j \theta_{ji} \exp \left( - \left( \frac{y_{ji}}{\alpha_{ji}} - \beta_{0j} y_i^0 - \sum_{k \in N_i \setminus j} \beta_{jk} y_{ki} \right) \right) + \lambda (y_i^0 + \sum_{j \in N_i} y_{ji} - y_i)$$

$$\frac{\partial L}{\partial y_i^0} = 0$$

$$0 = -\frac{\theta_i^0}{\alpha_i^0} \exp \left( - \left( \frac{y_i^0}{\alpha_i^0} - \sum_{j \in N_i} \beta_{0j} y_{ji} \right) \right) + \sum_{j \in N_i} \beta_{0j} w_j \theta_{ji} \exp \left( - \left( \frac{y_{ji}}{\alpha_{ji}} - \beta_{0j} y_i^0 - \sum_{k \in N_i \setminus j} \beta_{jk} y_{ki} \right) \right) + \lambda$$

$$\frac{\partial L}{\partial y_{ji}} = 0$$

$$= \frac{\theta_i^0}{\alpha_i^0} \beta_{0j} \exp \left( - \left( \frac{y_i^0}{\alpha_i^0} - \sum_{j \in N_i} \beta_{0j} y_{ji} \right) \right) + \frac{w_j \theta_{ji}}{\alpha_{ji}} \exp \left( - \left( \frac{y_{ji}}{\alpha_{ji}} - \beta_{0j} y_i^0 - \sum_{k \in N_i \setminus j} \beta_{jk} y_{ki} \right) \right) + \lambda$$

$$= \sum_{k \in N_i \setminus j} \beta_{jk} w_k \theta_{ki} \exp \left( - \left( \frac{y_{ki}}{\alpha_{ki}} - \beta_{0k} y_i^0 - \sum_{l \in N_i \setminus k} \beta_{lk} y_{li} \right) \right) + \lambda$$

Setting terms $A_0 = \frac{\theta_i^0}{\alpha_i^0} \beta_{0j} \exp \left( - \left( \frac{y_i^0}{\alpha_i^0} - \sum_{j \in N_i} \beta_{0j} y_{ji} \right) \right)$, $A_j = w_j \theta_{ji} \exp \left( - \left( \frac{y_{ji}}{\alpha_{ji}} - \beta_{0j} y_i^0 - \sum_{k \in N_i \setminus j} \beta_{jk} y_{ki} \right) \right)$, $j = 1, 2, \ldots, N_i$, the equations simplify to:

$$\lambda = \frac{1}{\alpha_i^0} A_0 - \sum_{j \in N_i} \beta_{0j} A_j$$

$$= \frac{1}{\alpha_{ji}} A_j - \beta_{0j} A_0 - \sum_{j \in N_i \setminus k} \beta_{kj} A_k$$

From the definitions of $A_0$ and $A_j$ terms, we get:

$$\frac{y_i^0}{\alpha_i^0} - \sum_{j \in N_i} \beta_{0j} y_{ji} = - \log(A_0 / \theta_i^0)$$

$$\frac{y_{ji}}{\alpha_{ji}} - \beta_{0j} y_i^0 - \sum_{k \in N_i \setminus j} \beta_{jk} y_{ki} = - \log(A_j / w_j \theta_{ji})$$

These sets of equations, together with the constraint, $y_i^0 + \sum_{j \in N_i} y_{ji} = y_i$, can help us in solving the problem. Define coefficient matrix $K \in \mathbb{R}^{(N_i+1) \times (N_i+1)}$ populated as follows. The diagonal comprises the
terms, \((1/\alpha_i, 1/\alpha_j, j \in N_i)\), the non-diagonal elements of the first row are \(-\{\beta_{ij}\}\), and those of the \(k^{th}\) row are \(-\{\beta_{kj}\}\). Note that the suppliers \(j \in N_i\) must be chosen in a well-defined order for the matrix \(K\) to be meaningful. Defining \(a_k = -\log(A_k/w_k\theta_{ki})\) and \(a = [a_k]\), the above equations can be written in a compact matrix form as: \(Ky = a, KA = \lambda l, and 1^T y = y_i\). Assuming \(K^{-1}\) exists, the disruption probability at optimum is then \(w_i = 1^T A = \lambda l^T K^{-1} 1\). Solving for \(\lambda\) in these equations, we get:

\[
y = K^{-1} a
\]

\[
\Rightarrow y = \sum_{k=1}^{N_i+1} [K^{-1}]_k a_k
\]

\[
= \sum_{k=1}^{N_i+1} [K^{-1}]_k (-\log(A_k/w_k\theta_{ki}))
\]

\[
= \sum_{k=1}^{N_i+1} [K^{-1}]_k (-\log(\lambda([K^{-1}1]_k)/w_k\theta_{ki}))
\]

\[
= \sum_{k=1}^{N_i+1} [K^{-1}]_k (-\log(\lambda) - \log([K^{-1}1]_k) + \log(w_k\theta_{ki}))
\]

\[
\Rightarrow y_i = \sum_{k=1}^{N_i+1} [K^{-1}]_k (-\log(\lambda) - \log([K^{-1}1]_k) + \log(w_k\theta_{ki}))
\]

\[
\Rightarrow \log(\lambda) \sum_{k=1}^{N_i+1} [K^{-1}]_k = -y_i + \sum_{k=1}^{N_i+1} [K^{-1}]_k (-\log([K^{-1}1]_k) + \log(w_k\theta_{ki}))
\]

\[
\Rightarrow \lambda = \exp \left( \frac{-y_i + \sum_{k=1}^{N_i+1} [K^{-1}]_k (-\log([K^{-1}1]_k) + \log(w_k\theta_{ki}))}{1^T K^{-1} 1} \right)
\]

\[
w_i = 1^T K^{-1} 1 \times \exp \left( \frac{-y_i + \sum_{k=1}^{N_i+1} [K^{-1}]_k (-\log([K^{-1}1]_k) + \log(w_k\theta_{ki}))}{1^T K^{-1} 1} \right)
\]

To solve for the decentralized equilibrium, the first-order conditions \(l \frac{\partial w}{\partial y_i} = -1\) yield, \(l^* w_i = 1^T K^{-1} 1\). Thus in the presence of co-dependence between investments, the term \(1^T K^{-1} 1\) plays the role of the in-degree. Indeed, the in-degree \(d_i^*\) can be recovered from \(1^T K^{-1} 1\) by setting the \(\beta\) terms to zero. However the intuition that the investments depend only on the extended local neighbourhood continues to hold: if firm \(i\) is a supplier to firm \(j\), then the disruption probability \(w_j^*\) depends only on \(l_i\) and \(1^T K^{-1} 1\), and both these quantities are intrinsic to firm \(i\).