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# Supply Chain Resilience - Epidemiological Characterization

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Disasters such as the Japanese tsunami and Thai floods in 2011 serve as a reminder about the extent to which a firm's operations are vulnerable to disruptions. The resulting losses arise not only through direct damage but also from disruption to partners beyond the adjacent tiers. Indeed, as per recent survey-based evidence, nearly 40% of the disruptions originate in tier 2 and beyond. In this paper, we propose an analytical framework to study the cascading of disruptions in supply-chain networks. We formalise the notion of *supply-chain resilience* as a measure of how quickly firms in the supply chain recover from a disruption. We also provide a comparative metric termed *relative vulnerability* to identify firms which are expected to suffer greater downtimes and consequently more losses due to disruptions. We illustrate these metrics on common supply-chain topologies and discuss the implications.

*Key words*: supply-chain disruptions; epidemiology; resilience; relative vulnerability
*History*: January 30, 2017

*"For want of a nail the shoe was lost,*
*for want of a shoe the horse was lost,*
*for want of a horse the King was lost,*
*for want of a King the battle was lost,*
*for want of a battle the kingdom was lost."*

– Old English folk rhyme

## 1. Introduction

Even seemingly trivial events can blow up to grave situations through a cascade of interlinked events. Similar cascading of disruptions is often observed in supply chains; for instance, Porsche was forced to halt production for two weeks in 2009, following the bankruptcy of a firm that manufactured the thread that went into its seat belts. The adverse impact of cascading disruptions is amplified dramatically in the context of disasters like the earthquake/tsunami that struck Japan in 2011. Economic losses from this event are estimated at $210 billion, of which only about $35 billion are insured losses. The disruption lasted in the supply-chain network for more than 6 months; supporting infrastructure such as ports and electricity supply were badly hampered, and multiple industries incurred substantial losses, e.g., automobile, electronics, steel, tire and rubber, chemicals, consumer goods, and even Disney theme parks (Airmic (2013)).

The trigger for disruptions are by no means limited to natural disasters. Supply chain disruptions could be caused by a myriad of different events, e.g., labor strikes, bankruptcy filing, industrial accidents, quality failures, etc. Modern day supply chains have proven to be particularly vulnerable to such disruptions, due to their global, interconnected[1] and complex nature. Thus, it is imperative that we refine our understanding of how the topology of the supply-chain network interacts with the various risk management strategies in place, to determine how quickly the entire network can rid itself of the disruption caused by a trigger event, and thereby curtail losses. In this paper, we introduce an analytical framework to help achieve that goal.

A key feature of disruption cascades, which makes their management particularly challenging, is that the events comprising the interlinked chain leading to losses, may occur at different stages or tiers in the network. We refer to this property of supply-chain disruptions as the *ripple effect*: disruptions to a firm's operations not only affect its immediate neighbors, e.g., suppliers or buyers, but often propagate along the supply chain to disrupt firms further away in the network. Indeed, as per a survey by the Business Continuity Institute, nearly 40% of all supply-chain disruptions originate from the second tier and beyond (Business Continuity Institute (2013)). The disruption in Japan in 2011 brought to light many instances of the ripple effect. One of them was the impact of Xirallic, a pigment used in metallic car paints (The Wall Street Journal (2011b)). As of 2011, the world's only plant producing Xirallic and catering to the entire global demand was located in Onahama in Japan, and this was severely damaged by the earthquake and accompanying tsunami. Consequently the production volume of firms like Toyota Motor Corporation, Nissan, Ford and Chrysler had to be cut by up to 20%. In the words of Dave Andrea, the then VP of the Original Equipment Suppliers Association (Financial Times (2011a)), "What vehicle manufacturers are finding are parts within parts within parts within parts that are sourced from a single-sourced Japanese manufacturer."

Apart from the ripple effect, the disaster in Japan exhibits a number of additional characteristics that are typical of disruption cascades in supply chains. In accordance with their importance, we highlight two such characteristics: the critical component property in the presence of complementary inputs; and the bidirectional propagation of disruptions. The former refers to the property

[1] An individual firm may process multiple products not necessarily limited to a single industry. This gives rise to interdependencies across the global supply-chain network.

that the shortage of any one input from a set of complementary inputs, is enough to completely stall a production line; this is true regardless of the size of the supplier, or the value of the input in short supply. Such a shutdown was caused by the closure of Hitachi's factory north of Tokyo which in 2011 made 60% of the world's supply of airflow sensors that are used to measure the amount of air coming into car engines. As a result, GM was forced to shut a plant in Louisiana and PSA Peugeot-Citroen had to cut back production at most of its European plants (The Wall Street Journal (2011a)).

While supplier failures invariably lead to disruptions, as with Xirallic and airflow sensors, disruptions can be bidirectional, i.e., problems at the buyer location can disrupt the supplier as well. The parts supplier GKN Plc. warned in March 2011 that it might cut the number of car components it makes because its Japanese customers were not expected to take deliveries (Reuters (2011)). Also, the Australian government estimated a loss of $2 billion from lost exports to Japan, which mainly commodities such as coal and iron ore (Yahoo! Finance (2011)). In general, disruptions at a firm, say a factory, will disrupt not only the buyer whose business depends on the factory's product, but also the supplier who provides raw material to the factory.

In the wake of the risk posed by the cascading or ripple effect, weakest-link property, and bidirectional propagation of disruptions, firms generally employ strategies to safeguard themselves from supply chain failure. Tomlin (2006) broadly classifies such strategies into two buckets: *mitigation* and *contingency* strategies. Mitigation strategies are preventive measures adopted by firms in order to thwart or delay the onset of disruptions, e.g., holding safety or buffer inventory. Contingency strategies are ex-post measures taken to recover from a disruption, e.g., sourcing from a "reliable" alternate or backup supplier. In the case of disruptions to a buyer, contractual safeguards such as "take-or-pay" can serve as a mitigation strategy, while a firm could sell its output to an alternate buyer (e.g., on the spot market) as a contingency measure.[2]

The interaction of network topology with the mitigation and contingency strategies in place determines how quickly a supply chain can recover from a disruption. The speed of recovery from a disruption can be viewed as a measure of *supply-chain resilience*. Over the years, this notion of resilience has proven to be notoriously elusive for various stakeholders – firms with large global supply chains, governments, and insurance providers. For instance, in the immediate aftermath of the tsunami in Japan, GM estimated losses of up to $1 billion, but later found the catastrophe's impact on its production to be minimal, thereby suggesting that GM may have underestimated the resilience of their supply chain. On the other hand, a number of firms overestimated resilience

[2] A take-or-pay clause in a contract stipulates that the buyer either takes the product from the supplier or pays the supplier a penalty.

and did not anticipate the kind of business interruption that ultimately occurred. Our first contribution is to formalize the notion of supply-chain resilience: given an arbitrary network topology, information about the deployed risk management strategies, and a trigger event for the disruption, we propose a summary metric that captures how quickly all the nodes in the supply-chain network can be rid of disruptions.

Another question of managerial interest is to determine the relative vulnerability of different firms in the supply-chain network, i.e., their relative likelihood of remaining disrupted in the long run. Such information would help managers prioritise where in their supply chain to invest limited financial resources in order to mitigate disruption risk. Further, it could potentially serve as an aid to providers of supply chain insurance for determining the appropriate risk-based premium to charge a client firm depending on its risk management practices and location in the network. Therefore, our second contribution is to propose a measure of relative vulnerability that is intimately linked to our metric for supply-chain resilience.

Finally, we discuss the role of topology in propagating disruptions, by comparing and contrasting the resilience of different supply chain topologies to derive insights on the relationship between network structure and severity of disruption cascades.

To summarize, our contributions in this paper are three-fold:

- We put forth a framework to describe cascading disruptions in a supply chain network.
- For the purpose of analytical tractability, we come up with an epidemiological model which bounds the losses incurred by firms in the base framework.
- We use the second model to generate insights about the relationship between supply chain resilience, network structure, and the position of individual firms in the network.

The paper is organized as follows. In the next section, we contrast our work with the literature on several similar problems involving diffusion in networks and enunciate the unique features of our problem and methodology. In Section 3, we describe our model, and in Section 4 we provide analytical bounds on how quickly a supply-chain network recovers completely from a disruption. In Section 5 we formalize the notion of supply-chain resilience and relative vulnerability, and derive these metrics for some common topologies. We conclude in Section 6.

## 2. Literature Survey

Our work is related to two major streams of literature: one, the study of disruption risk in operations and two, the study of mechanisms of diffusion in networks. In this section we survey the operations literature dealing with disruptions in supply chains and also subsequently analyse how the study of network effects has taken shape in other areas like finance, economics, physics, epidemiology, etc.

*Literature on disruptions in supply chains:* The problem of supply-chain disruptions and their impact on business operations has been well-studied in the literature for several years now. There are a number of qualitative papers that have studied the problem, e.g., Kleindorfer and Saad (2005) and Netessine (2009). Kleindorfer and Saad (2005) analyse the effect of sudden disruptions on a supply chain and broadly identify two strategies to counter the impact. One is to reduce the frequency and severity of risks at firm-level and across the supply chain. The second measure is to increase the capacity of supply chain participants suitably to absorb the impact of disruptions. Netessine (2009) calls for new modelling techniques to reflect network effects in supply-chain disruptions.

One of the first papers to analytically study unreliable supply was Meyer et al. (1979), and failure of a supply in a two-supplier setting was first studied by Parlar and Perry (1996). More recently, DeCroix (2013) considers an assembly system and studies the impact of disruptions on optimal orders in this network. These papers have typically focused on the optimal ordering policy in the face of unreliable supply. Other papers such as Tomlin (2006) study strategies to counter supply-chain disruptions. Specifically, the paper analyzes the worth of mitigation strategies like dual sourcing and inventory management and that of contingency strategies like rerouting in the event of a disaster. The authors characterize different scenarios under which each of those strategies may be optimal.

Another popular strategy to manage the risk from disruptions is to purchase insurance. The Thai floods (Financial Times (2011b)) and Japanese tsunami of 2011 (CBC News (2011)) are estimated to have cost the insurance industry losses amounting to US$10 billion and US$35 billion respectively. Insurance for firms against losses due to supply-chain disruptions is provided by business interruption (BI) insurance and contingent business interruptions (CBI) insurance. While both cover losses due to the inability of a firm to conduct operations as normal, the former covers losses arising from direct damage to a firm's facilities, while the latter covers losses triggered by physical damage at a supplier or client's facility. There are certain specific situations under which a firm is eligible to claim CBI; we list three such considerations. Firstly, there must be *physical damage* to its immediate supplier or buyer. A firm cannot claim CBI for a disruption to its second-tier supplier, which has in turn caused operational disruptions to the immediate supplier. Neither can a firm claim CBI for damages that do not involve physical damage at a supplier or client's facility. Secondly, the peril must be covered in the insurance scheme; the insurance does not apply to any unforeseen peril. Thirdly, CBI covers business losses only for a fixed amount of time after the disruptions. Dong and Tomlin (2012) study the interplay between operations and insurance in detail and consider the efficacy of business interruption (BI) insurance vis-a-vis the usual operations strategies like inventory management and emergency sourcing.

In a slightly different setting, Simchi-Levi and Wei (2012) relate the issue of supply disruptions to process flexibility. Process flexibility, as defined by Jordan and Graves (1995), considers a single firm manufacturing multiple products and different firms. Process flexibility relates the ability to manufacture different products in different plants. A process if *full-flexible* if every product can be produced in any of the available plants. Simchi-Levi et al. (2013) argue that having multiple plants producing a product is essentially akin to having multiple supply sources in a supply chain. They then compare the relative merits of process flexibility to holding strategic inventory. In summary, while there is a lot of work on disruption risk management in supply chains, the existing papers generally restrict attention to simple topologies and almost never dwell on the ripple effect of supply-chain disruptions, or on the interplay between network topologies and disruption risk.

A notable exception to this observation is the recent work by Simchi-Levi et al. (2014), where the authors undertake the task of risk mitigation for Ford's supply chain. The authors use *time to recovery* (TTR) of a facility, which they define as the time for a facility to reach full functionality after a disruption, to characterise interdependencies in the supply chain. They then compute the *risk exposure index* (REI), which is an index denoting vulnerability of various facilities in the supply chain. However the authors focus on Ford's internal supply-chain network comprising its raw material manufacturers, distributors and retailers. Moreover the approach taken in the paper is essentially computational. In this paper we aim to provide a theoretical framework towards understanding resilience and relative vulnerabilities of any arbitrary supply-chain network and understand the relationship between supply chain topology and the severity of the ripple effect.

*Literature on cascades in networks:* We find that our problem of cascading disruptions in supply chains is similar to problems in many other areas dealing with propagation of a particular factor (analogous to disruption in our problem) within a network. Table 1 presents a broad overview of such problems possessing a setting similar to ours. In all of these problems, there exists a network, an initial seed, a mechanism through which the seed propagates in the network and another mechanism by which this propagation is impeded. We term the propagation mechanism as *disruption* and the impeding mechanism as *recovery*, taking cues from epidemiological literature.

Consider a financial network, for example, with a set of banks borrowing from one another and investing in some common assets. The interdependency between banks is determined by the extent of interbank lending. Of late, several papers have focused on modeling systemic risk in financial networks, which is the risk of default for all firms within a network in the event of a shock to a common asset (to name a few, Allen et al. (2012), Glasserman and Young (2014), Acemoglu et al. (2013), Elliott et al. (forthcoming)). In this case, infection between two banks could be construed as the rate at which one bank defaults contingent on the failure of its neighbour. However, papers dealing withsda systemic risk do not consider a recovery mechanism for banks after the shocks

| Theme | Initial condition | Infection | Recovery |
|---|---|---|---|
| Contagion spread in financial networks | A network of banks holding common assets and interbank debts; prices of some assets receive shocks | Defaulting of a bank which cannot fulfill its debt obligation causes losses to other banks which have lent to it. | No recovery is available, but analysis is useful to get an idea about vulnerability of different banks to defaulting |
| Information diffusion in social networks | Information (or a new product) is made available to a few people in a society | 1. Nodes buying the product due to a recommendation can be said to be infected. Nodes can be in one of three states: susceptible (prone to buy), infected (bought) and immune (not buying).<br>2. The infection rate between two nodes is proportional to the degree of mutual familiarity or trust. | No recovery; in equilibrium, either nodes become infected (share the information or buy the product) or stay immune (choose not to share or buy). |
| Aggregate volatility in a network of economies | A network of sectors of economies depending on one another's output; one or more sectors faces a shock to its output | Shock propagation causing an aggregate volatility in the total output | No recovery mechanism, as in financial networks. |
| Resilience of a population to the spread of epidemic | Epidemic spreading through a group of people in a society starting with a few sick individuals | 1. Infection rate between every pair for individuals<br>2. Rate of infection is proportional to extent of contact | 1. Recovery at a rate that is individual-specific<br>2. Infected individuals could either become immune to the infection post-recovery (SIR), or remain susceptible yet again (SIS). |
| Resilience of supply-chain networks to propagation of disruptions | A group of firms in a supply chain where one or more are disrupted initially. | 1. Node-based recovery, i.e., infection propagation happens along every link in the supply chain.<br>2. A firm is infected if at least one of its incoming links is infected. | 1. Link-based recovery, i.e., there exists a mechanism for recovery of every infected link.<br>2. Recovery of a firm results in all its outgoing links recovering. |

**Table 1 Summary of literature involving network contagion spanning multiple fields**

occur, instead focusing on the effect of the network structure and the extent of interbank lending on the defaulting process.

In parallel, there have been a few papers in economics dealing with the impact of network structure. In Costinot et al. (2013), the authors develop an elementary theory involving global supply chains. There is a sequence of countries $c_1, c_2, \ldots, c_n$, and a product moves along a supply chain that moves through the $n$ different countries. Each of those countries has a rate of failure: a country $c$ has a rate of failure $\lambda_c$ with which a mistake can occur in the product manufacturing at that stage in the supply chain. If a mistake happens anywhere in the supply chain, the product is lost totally. Besides, there is a wage rate and an export price that is distinct to different countries. The authors then proceed to analyse the value of technological improvement (i.e., being less prone to making mistakes) for countries at different stages in the supply chain. On a related note, Acemoglu et al. (2012) analyse the role of network structure in propagating aggregate volatility. A group of economies form a network based on the dependencies on each other's output. There arise shocks that affect the output of an individual firm. If economies are independent of one another, we expect the aggregate volatility to decrease as $\mathcal{O}\left(\sqrt{\frac{1}{n}}\right)$ with the number of firms economies $n$. However the authors show that the network structure plays a key role in determining whether to ignore the contribution of individual economy-level shocks to the overall output. Another work in economic theory related to network studies is the paper by Kranton and Minehart (2000) where the authors study pricing and allocation amongst a group of sellers and buyers. Every buyer (seller) has a select set of seller (buyer) with whom he can transact, thus the permissible network connections between buyers and sellers is limited. The network connections lead to a bargaining problem and the paper theoretically derives solutions for optimal allocation and pricing problem. In each of these problems, while the role of network structure plays an important role in the problem, there is no analogue to the infection and recovery mechanisms we put forth for the study of cascades of supply-chain disruptions. Moreover, these papers involve an equilibrium treatment, while the problem of cascading disruptions in a network is a transient phenomenon.

A more closely related problem is that of an epidemic spreading among a population of individuals. The epidemic begins with a single or a few infected individuals; assuming the entire population is susceptible, the infection propagates through contagion. The speed at which individuals recover depends on their immunity and medication. While the epidemic propagation model closely resembles our model for supply-chain disruptions, we identify one key difference: the recovery process in epidemics is individual-dependent. That is, irrespective of whom the infection is contracted from, an individual always recovers at a rate specific to him/her, based on one's inbuilt immunity and medication taken. In contrast, recovery for a firm in a supply chain is link-specific - a car manufacturer could be disrupted either due to the failure of the seat supplier or the tyre supplier, but the recovery after disruption has to be different in each case, based on feasible contingency strategies, like the availability of a temporary supplier.

Two distinct models of epidemic propagation are considered in the literature: the SIS (Susceptible-Infected-Susceptible) model and the SIR (Susceptible-Infected-Recovered) model. The SIS model is one where the people who recover from an infection stand prone to an infection once again. On the other hand, in an SIR model, recovered individuals develop immunity towards the disease and are not infected for a second time. Under either of these models, any infection starting from a set of nodes will heal entirely with probability 1 in any network with finite number of individuals. There has been a lot of research in mathematics, physics and epidemiology on the propagation of epidemics on infinite graphs. Results from percolation theory indicate the presence of an epidemic threshold in many networks: when the ratio of infection rate to recovery rate is below a particular threshold, the infection dies out in any large network, and if the ratio is larger than a threshold, the infection propagates and persists in the network for a long time. There has also been research on the role of network topology in facilitating infection propagation. For asymptotically large graphs, one of the first papers in this direction was Pastor-Satorras and Vespignani (2001), who specifically study a scale-free network, i.e., a graph whose node-degree distribution is distributed as a power-law function ($\propto x^{-\alpha},\ \alpha > 1$, where $x$ is the number of nodes.) They find that the epidemic threshold is absent on a wide range of scale-free networks for different values of $\alpha$. Newman (2002) consider an SIR model, and provide analytic expressions for epidemic threshold and size on networks with arbitrary degree distributions. The SIS model has also been studied by the mathematics community under the title of *contact process*. Early results from contact processes on $d$-dimensional integer lattices have been presented in Liggett (1999). Further asymptotic results on epidemic threshold for SIS mode of propagation have been presented for random power-law graphs in Berger et al. (2005) and Chatterjee and Durrett (2009) and for the complete graph in Peterson (2011).

For SIS propagation on finite graphs, a key result is provided by Ganesh et al. (2005), where the authors prove that if the ratio of infection to recovery rates is lesser than the spectral radius (the largest eigenvalue of the adjacency matrix), then the mean lifetime of the epidemic is of the order $\mathcal{O}(\log(n))$. In a finite network with $N$ nodes, the exact dynamics of the SIS process can be described by a continuous time Markov chain (CTMC) with $2^N$ states. Chakrabarti et al. (2008) and Van Mieghem et al. (2009) provide a comparison between a reduced $N$-differential equations system and the exact $2^N$-state CTMC system. Van Mieghem et al. (2009) enunciate a differential equation model, called the $N$-intertwined model to describe a SIS propagation. This reduced $N$-state model simplifies the Markov system using mean-field techniques and approximates the $2^N$-state CTMC model. The authors also prove the existence of the epidemic threshold, which is equal to the inverse of the spectral radius of the adjacency matrix and prove that even in a finite network, infections could persist for an arbitrarily long time before all nodes recover completely, if the ratio of the infection rate to the recovery rate is greater than the epidemic threshold. The infection and recovery rates are assumed to be homogeneous in these papers. However, since both the infection and recovery processes in supply chain disruption cascades are link-based, we find such existing epidemic models are not directly suitable to describe them. Yet, as we show in Section 4, it is possible to bound the speed of recovery of a disrupted supply-chain network by SIS-type systems with appropriately chosen parameters.

In the next section, we put forth the supply-chain-disruption cascade (SCDC) model to explain the ripple effect of supply-chain disruptions.

# 3. Supply chain disruptions cascade (SCDC) model

## 3.1. Model Description

The supply-chain-disruption cascade (SCDC) model is essentially an epidemiology-inspired model to describe the ripple effect of disruptions in supply chains. We consider a directed graph $G(V,E)$ of $N$ nodes to represent a supply chain, with nodes representing firms, and the presence of an edge denoting the existence of a supplier or buyer relationship between a pair of firms.[3] As edges in $G$ could represent both supplier and buyer relationships, every directed edge in $E$ has a counterpart in the opposite direction, i.e., $(i,j) \in E \implies (j,i) \in E \;\forall\; i,j \in V$. Since the ripple effect arises from the interaction of multiple firms, each with its idiosyncratic risk-management strategies, the key modelling challenge is to come up with a tractable framework at an appropriate level of abstraction. We recall Tomlin (2006)'s classification of mitigation and contingency strategies described in Section 1. Depending on the mitigation strategy, we model an *propagation rate* along the edge between two firms, which is the rate at which the destination firm gets disrupted when the source firm is down. For example, the amount of a inventory held by a firm for a particular input could determine the infection rate from the corresponding supplier. On the other hand, the infection rate from a buyer could be determined by contractual obligations. For example, the buyer could have agreed to continue receiving supply for a fixed period even after it is disrupted. Similarly, corresponding to the contingency strategy, there exists a *recovery rate*, which is the rate at which the destination node recovers through contingent means when the original supplier or buyer is disrupted. In the case of a supply disruption, a contingency strategy is to reroute supply temporarily from an alternative and reliable supplier. For buyer disruptions, selling the product at the spot market is a possible contingency tactic. Both the propagation and recovery processes are specific to the relationship between two firms.

[3] If both nodes belong to a single firm, e.g, a factory and a warehouse, an edge is used to represent either a sender or receiver relationship between the two nodes.

For every link $(i, j)$, we define $\lambda_{ij}$ as the propagation rate, or the rate at which $j$ is disrupted once $i$ is disrupted, and $\mu_{ij}$ as the recovery rate, or the rate at which $j$ recovers through contingent means after it is disrupted because of $i$. We note that unlike the SIS models reviewed in Section 2, the recovery rates in this model are edge-specific. Both the propagation and recovery processes are random and governed by stochasticity in demand and lead time. For simplicity, we assume that all infection and recovery rates across different links are Poisson and are independent of one another.

For the initial conditions, we assume that no firm in the network is disrupted until time $t = 0$, when a subset of firms $\mathcal{I} \subseteq \{1, 2, \ldots, N\}$ get disrupted, and each of these firms recover independently at rate $\mu_i,\ i \in \mathcal{I}$.

### 3.2. Key Assumptions

We now highlight four assumptions in our model that help us to capture reality approximately, while still paving the way for a tractable abstraction. First, the contingent strategy is a temporary measure and firms return to sourcing or buying from the original supplier or buyer as soon as the latter recover from their disruptions. Second, since the contingency strategy is a temporary solution, we assume that it is perfectly reliable and free from the threat of disruptions. In other words, firms do not face threat of cascade along an edge, as long as they are dependent on contingent supply. Third, we only capture 0/1 disruptions at nodes, and partial disruptions, wherein firms cut their capacity without being completely disrupted, are not accounted for. A firm is said to be *disrupted* if it is unable to meet the needs of its clients; the firm stops purchasing from its suppliers and supplying to its buyers. Fourth, consistent with our setting of complementary inputs and large losses, we assume that the failure of even one of a firm's suppliers or buyers puts it at the risk of getting disrupted by the corresponding supplier or buyer.

### 3.3. Edge and node states

The SCDC model captures a *transient* system in which disruptions initially affect firms $i \in \mathcal{I}$ at $t = 0$, then cascade through the supply chain, and eventually cease to exist after all firms in the network recover. Disasters which trigger supply-chain disruptions are referred to as *unknown-unknown* by Simchi-Levi (2010), as accurate estimation of their probabilities is impossible due to lack of data. Hence we do not assume prior probabilities on these high-risk, low-probability events and instead study cascading of disruptions conditioned on the occurrence of an initial trigger event. The state transitions which govern the ripple effect in the SCDC model can be described as follows. At any point in time, there are candidate links and nodes in the network for propagation and recovery. Any link between a disrupted firm and a *healthy* (i.e., undisrupted) firm is a candidate link for propagation. Any link $(i, j)$ with both $i$ and $j$ disrupted is a candidate link for recovery. The originally disrupted firms in $\mathcal{I}$ are candidate nodes for recovery. When a node $v \in V$ recovers, the supplier and buyer nodes disrupted by $v$ recover immediately. Also suppliers and buyers of $v$ that were disrupted originally but managed to recover through contingency means abandon the contingency plan and revert to supplying or buying from $v$.

To describe these dynamics, we define node and edge states. Let $v_i(t)$ denote the state of the firm $i$ at time $t$, and a node can be in one of two states: $v_i(t) = 1$ if the firm $i$ is down as a result of supply-chain disruptions at $t$ and 0 otherwise. In a similar vein, we define $e_{ij}(t)$ to denote the state of the edge $(i, j)$ at time $t$. An edge can be in one of three states to be defined as follows.

$$e_{ij}(t) := \begin{cases} 0, & \text{when no disruption has propagated from firm } i \text{ to } j, \\ 1, & \text{if disruption has propagated from } i \text{ to } j \text{ and } v_i(t) = v_j(t) = 1, \\ 2, & \text{if disruption has propagated from } i \text{ to } j \text{, but } j \text{ has recovered through contingency means while } i \text{ remains disrupted (in this case, } v_i(t) = 1 \text{ and } v_j(t) = 0). \end{cases}$$

To clarify the three edge states, note that $e_{ij}(t) = 1$ for the period of time after firm $i$ has disrupted firm $j$ and both are down. When the state of the edge $e_{ij}(t) = 2$, then firm $i$ remains disrupted, firm $j$ has recovered through contingency means and disruptions cannot cascade from firm $i$ to firm $j$. When $i$ recovers, $e_{ij}(t)$ is reset to 0.

### 3.4. Transition Rates

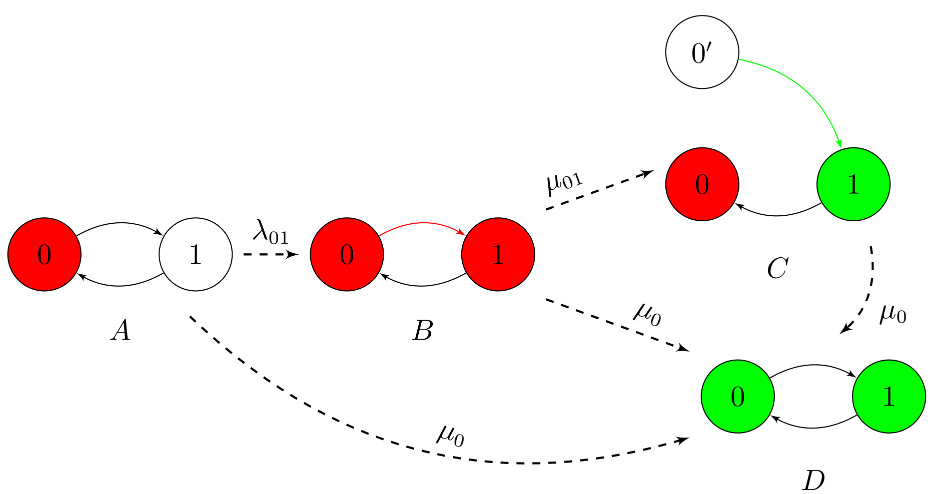


**Figure 1 CTMC corresponding to the SCDC model in a supply chain of two firms 0 and 1, with firm 0 disrupted at time $t = 0$. Node $i$ is coloured in a state in red if $v_i = 1$ when the process is in that state. Edge $(i, j)$ is coloured in black, red and green when $e_{ij} = 0$, 1 and 2 respectively.**

To show our model in action, we consider a simple two node supply chain in Figure 1, with firm 0 is disrupted at time $t = 0$. There are four states in the resulting continuous time Markov chain

(CTMC), represented as A, B, C and D. A is the initial state; in A, $v_0 = 1$, $v_1 = 0$ and $e_{01} = 0$. The CTMC is in state B when both firms 0 and 1 are disrupted; in B, $v_0 = 1$, $v_1 = 1$ and $e_{01} = 1$. C is the state where firm 0 remains disrupted but firm 1 recovers through a contingency supply from firm $0'$. In C, $v_0 = 1$, $v_1 = 0$ and $e_{01} = 2$. In state D, firm 0 also recovers from the original disruption; in D, $v_0 = 0$, $v_1 = 0$ and $e_{01} = 0$. There are two points that are worth noting in regard to this CTMC. Firstly, the states in the CTMC and the corresponding transition rates are *defined* by the original graph and the initially disrupted nodes. A change in the set of initially disrupted nodes $\mathcal{I}$ or a change in the topology $G$ will define a new state space for the CTMC to evolve. Secondly, the CTMC is transient, with a single absorbing state in D.

CTMC representation becomes cumbersome for even small graphs, as the number of allowed states can be prohibitively large. Instead, the transition in our model can be compactly represented using edge and node transition rates. We consider the dynamics of disruption propagation and recovery in the following discussion.

- **Disruption propagation**: The edge $e_{ij}$ can transition from 0 to 1 at the rate $\lambda_{ij}$ when firm $i$ is disrupted, but firm $j$ is not; hence, the effective rate of transition is $\lambda_{ij}\mathbb{1}_{\{v_i=1\}}\mathbb{1}_{\{v_j=0\}}$. Similarly, when firm $i$ is undisrupted, a disruption can arrive from any of its disrupted suppliers, and since these arrivals are independent, the rate at which $v_i(t)$ changes from 0 to 1 is the Poisson sum of the corresponding disruption rates.
- **Recovery**: There are two kinds of recovery in our model. First, the firms in $\mathcal{I}$ which were originally disrupted recover at rates $\mu_i, i \in \mathcal{I}$. Second, for disrupted firms not in $\mathcal{I}$, two kinds of recovery is possible - either their supplier recovers, or they find a contingent recovery mechanism. The two processes are independent, and for a firm $i$ disrupted by $j$, occur at rates $R_j$ and $\mu_{ji}$ respectively, where $R_j$ can be recursively defined in terms of the recovery rates based on the actual source of the disruption (see SCDC below) . Hence $e_{ji}$ shifts from 1 to 0 and 2 to 0 at rate $R_j$, and from 1 to 2 at rate $\mu_{ji}$. On the other hand, $v_i$ transitions from 1 to 0 at a rate $R_j + \mu_{ji}$.

The foregoing discussion can be summarized as follows:

$$e_{ij}(t): \begin{cases} 0 \to 1, & \text{at rate } \lambda_{ij}\mathbb{1}_{\{v_i=1\}}\mathbb{1}_{\{v_j=0\}}, \\ 1 \to 2, & \text{at rate } \mu_{ij}, \\ 1 \to 0, & \text{at rate } R_i, \\ 2 \to 0, & \text{at rate } R_i, \text{ and,} \end{cases} \tag{SCDC-edge}$$

$$v_i(t): \begin{cases} 0 \to 1, \text{ at rate } \sum_j \lambda_{ji}\mathbb{1}_{\{v_j=1\}}\mathbb{1}_{\{e_{ji}=0\}}, \\ 1 \to 0, \text{ at rate } R_i \text{ for } i \notin \mathcal{I} \text{ and } \mu_i \text{ for } i \in \mathcal{I}, \end{cases}$$
$$\text{where } R_i = \sum_k \mathbb{1}_{e_{ki}=1}(\mu_{ki} + R_k), \text{ if } i \notin \mathcal{I}, \text{ and } R_i = \mu_i, \text{ for } i \in \mathcal{I}. \tag{SCDC}$$

## 4. Analysis: Network recovery

In this section, we undertake a study to characterise how quickly firms in a supply chain will recover from an initial disruption. As we noted in the example in the previous section, the state transitions involved in the SCDC model can be represented as a continuous time Markov chain (CTMC). However, if there are $M$ edges and $N$ vertices in the network, at any instant in time, the state of every node and edge has to be tracked. The total number of states in such a CTMC is $2^N 3^M$, and hence evaluating our model through a CTMC whose state is the vector of states of all nodes and edges in the graph is computationally intractable.

We approximate our SCDC model in order to bound the speed of convergence to recovery of all firms in a supply chain. To define the rate of convergence to recovery precisely, let us consider $\mathbf{p}^{SCDC}$, the vector of probabilities of each of the $N$ firms being disrupted at time $t$ when evolving under the transition rates specified by the SCDC model, i.e.,

$$\mathbf{p}^{SCDC} = [\mathbb{P}(v_1(t)=1) \quad \mathbb{P}(v_2(t)=1) \quad \ldots \quad \mathbb{P}(v_N(t)=1)]^\top$$

Under the evolution specified by the SCDC model, we are interested in the convergence to the equilibrium state when $v_i = 0$ for all nodes $i$; we term this equilibrium as *network recovery.*

Definition 1 (Network Recovery). A network of firms is said to have *recovered* when all firms are rid of disruptions, that is $v_i = 0 \ \forall i \ \in V$.

Network recovery is a function of the propagation and recovery rates and the topology of the graph $G$. We need to identify conditions on the rates and the topology that ensure network recovery will materialise. In the event all firms in the network recover from disruptions, the probability vector $\mathbf{p}^{SCDC}$ starting at $t=0$ from a non-zero value, decays to zero. If it is possible to bound the decay of $\mathbf{p}^{SCDC}$ using an exponential function, we term the equilibrium state $\mathbf{p}^{SCDC} = \mathbf{0}$ to be *exponentially stable.* If network recovery is possible and the equilibrium is exponentially stable, we define the rate of convergence to recovery $\rho_{SCDC}$ to be the exponential rate at which the norm of the probability vector $||\mathbf{p}^{SCDC}||$ decays to 0.

Definition 2 (Exponential stability). An equilibrium point $\mathbf{p}^{SCDC} = \mathbf{0}$ is exponentially stable if there exist positive constants $\kappa$, and $c$ such that $||\mathbf{p}(t)|| < ||\mathbf{p}(0)||e^{-\kappa t}, \ \forall ||\mathbf{p}(0)|| < c$ for any initial state $\mathbf{p}(0)$.

Definition 3 (Exponentially bounded decay). When the equilibrium point $\mathbf{p}^{SCDC} = \mathbf{0}$ is exponentially stable, the norm $||\mathbf{p}^{SCDC}||$ is said to have an exponentially bounded decay if there exist positive constant $\rho^{SCDC}$ such that

$$||\mathbf{p}(t)|| \leq ||\mathbf{p}(0)||e^{-\rho_{SCDC} t}$$

for any initial state $\mathbf{p}(0)$. In that case, we say the rate of convergence to network recovery is equal to $\rho^{SCDC}$.

In the next part, we first describe the SIS model of epidemic propagation. We bound the rate of convergence of the probabilities in SCDC model using the corresponding probabilities in the SIS model.

*SIS model of epidemic propagation:* A standard model of epidemic propagation in epidemiological literature is the Susceptible-Infected-Susceptible (SIS) model. In this model, a graph is considered with vertices representing people and edges for the potential contact between them. Between every pair of individuals, there exists an infection rate which is the rate at which an infection in the source node spreads to the destination node. Every node has its own recovery rate, which is idiosyncratic and independent of where the infection was contracted from. Recovered nodes become susceptible to infection once again, hence the name S-I-S. A major difference between SCDC model and the SIS model is in the recovery process. In supply-chain disruptions, recovery of a firm depends crucially on where the disruption originated from, or in other words, which of its suppliers or buyers caused the disruption. Thus recovery is a link-based trait and is not node-based, as is the case in SIS model. The SIS model can be represented purely using nodal transitions and unlike the SCDC model, there is no interaction between node and edge processes. Apart from the presence of node-specific recovery, there are a couple of downsides to the SIS model, which makes it unsuitable for capturing the node-edge interactions present in SCDC model. Firstly, the recovery process in SIS model is node-based and independent of other recovery and infection processes. Suppose firm A is disrupted and this creates a disruption to firm B at rate $\lambda_{AB}$. If A recovers, then we would expect B to recover soon, but in this model B recovers independently at a rate $\mu_B$. Secondly, the SIS model is not suitable to capture contingent recovery. Suppose firm A is disrupted, it disrupts firm B and firm B recovers through contingent means by opting for a reliable third-party supplier. B is immune from further disruptions from A unless A recovers, in which case B switches to A, abandoning the contingency strategy. However, since link states are not maintained in the SIS model, firm A could still potentially disrupt B at a rate $\lambda_{AB}$ as long as A is down and B is up.

However, as we will show in this section, the SIS model could be used to provide useful bounds for the rate of recovery from disruptions in SCDC model. To this end, retaining our notation, we define a new model, SIS, such that the probability of a firm being disrupted at $t$ under the evolution specified by the SCDC model is bounded above by the corresponding probability in the SIS model. We define SIS through the following transition rates.

$$\begin{aligned} v_i(t) &: \begin{cases} 0 \to 1, \text{at rate } \mathbb{E}\left[\sum_j \lambda_{ji} \mathbb{1}_{\{v_j=1\}}\right] \\ 1 \to 0, \text{at rate } \min_j \mu_{ji} \text{ for } i \notin \mathcal{I} \text{ and } \mu_i \text{ for } i \in \mathcal{I} \end{cases} \\ e_{ij}(t) &= v_j(t),\ \forall\ t \geq 0 \text{ and } \forall\ (i,j) \in E \end{aligned} \tag{SIS}$$

By defining the transition rates as above, we have effectively gotten rid of the dependency of edge states on nodal transitions that we encountered in the SCDC model. That is, the recovery rates in either of these models are some function of $\mu_{ij}$s and $\mu_i$s, but do not depend on node states $v_i(t)$ or edge states $e_{ij}(t)$. The intuition for the choice of these rates is as follows. Since the probability of disruption in the SIS model must upper bound the corresponding probability in the SCDC model, we have to choose the worst possible recovery rates for nodes. If a node $i$ is disrupted, in the worst case, it can recover at the minimum rate $\min_j \mu_{ji}$. In other words, if $j^* = \arg\min_j \mu_{ji}$, then from the point of view of firm $i$, finding a contingency strategy will be most difficult when it suffers a disruption from firm $j^*$. Hence $\min_j \mu_{ji}$ can serve as the worst-case node-specific recovery rate for firm $i$. Also, by defining $e_{ji} = 1$ whenever $v_i = 1$ we have ensured that the propagation rates are equal in both models.

The following theorem which is the main result of this section, states that the probability that a firm will remain disrupted at any point in time in the SCDC model is bounded by the corresponding probability in SIS model. We define probability vector $\mathbf{p}^{SIS}(t)$ in the same way as $\mathbf{p}^{SCDC}(t)$, i.e., the vector of probabilities of nodes being disrupted at time $t$ when evolving under the rates specified by SIS model. Also, for $i \notin \mathcal{I}$, let $\mu_i^U = \min_j \mu_{ji}$ and for $i \in \mathcal{I}$, $\mu_i^U = \mu_i$, and define the matrix,

$$\mathbf{R}_U = \begin{bmatrix} -\mu_1^U & \lambda_{21} & \cdots & \lambda_{N1} \\ \lambda_{12} & -\mu_2^U & \cdots & \lambda_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1N} & \lambda_{2N} & \cdots & -\mu_N^U \end{bmatrix}.$$

The proof of Theorem 1 uses a stochastic coupling argument and is presented in the appendix.

Theorem 1. *Let $\mathbf{v}^{SIS}(t)$ and $\mathbf{v}^{SCDC}(t) \in \{0,1\}^N$ be the state vectors of the $N$ nodes at time $t$ when evolving under the SIS and SCDC models respectively. The initial conditions are same for both vectors ($\mathbf{v}^{SIS}(0) = \mathbf{v}^{SCDC}(0)$) and $v_i^{SIS}(0) = 1$ for firms $i \in \mathcal{I}$ and $v_i^{SIS}(0) = 0$ else. Then,*

1. *$\mathbb{P}(v_i^{SIS}(t) = 1) \geq \mathbb{P}(v_i^{SCDC}(t) = 1)$ for all $i$ and $t > 0$. Consequently, $||\mathbf{p}^{SCDC}(t)|| \leq ||\mathbf{p}^{SIS}(t)||$.*
2. *$||\mathbf{p}^{SIS}(t)|| \leq ||\mathbf{p}^{SIS}(0)|| e^{\Re(\lambda_{max}(\mathbf{R}_U))t}$.*

The following corollary provides a bound on the exponent of convergence in the SCDC model, as a function of the transition rates determined by the risk mitigation strategies of firms.

Corollary 1. *A necessary condition for system recovery is that the real parts of all eigenvalues of $\mathbf{R}_U$ should be negative. Moreover, if $\Re(\lambda_{max}(\mathbf{R}_U)) < 0$ and if $\mathbf{p}^{SCDC}$ has an exponentially bounded decay, then the following bound holds:*

$$-\Re(\lambda_{max}(\mathbf{R}_U)) \leq \rho_{SCDC}, \tag{1}$$

*where $\lambda_{max}$ and $\lambda_{min}$ are eigenvalues with maximum and minimum real parts of the respective matrices and $\Re(\cdot)$ denotes the real part of a complex argument.*

## 5. Resilience and relative vulnerability of common supply chain topologies

Suppose a social planner is provided with data on interconnections between a set of firms and their propagation and recovery rates in the event of disruptions. We identify two key performance measures from the social planner's perspective - the resilience of the supply chain, which is a measure of the ability of the overall supply chain to withstand the spread of disruptions, and the relative vulnerabilities of different firms to disruptions, which are in effect, weights assigned to firms based on their likelihood to be down due to disruptions. Ideally we would like both these measures to be independent of the source of disruptions.

We saw in the previous section that the most compact characterisation of the SCDC model is using the bounds on the rate of convergence to recovery. While evaluating performance measures, a social planner is likely to be interested in minimising losses in the worst possible scenario. In a bid to derive these measures, we focus on the SIS model in this section, which represents the worst case disruption probabilities. To define the resilience, we recall the dynamics of probabilities of disruption in the SIS model:

$$\frac{d\mathbf{p}^{SIS}(t)}{dt} = \mathbf{R}_U \mathbf{p}^{SIS} - \text{diag}(\mathbf{\Lambda}\mathbf{p}^{SIS})\mathbf{p}^{SIS}$$

DEFINITION 4. The **resilience** $\xi$ of a supply chain topology specified by the graph $G(V,E)$, the propagation and recovery rates $\lambda_{ij}$ and $\mu_{ij}$, $i,j=1,2,\ldots,N$ is defined as the lower bound in Equation 1, i.e., $\xi = -\Re(\lambda_{max}(\mathbf{R}_U))$.

There are a few important points to note about the resilience metric, which is essentially the worst case rate of convergence to recovery. Firstly, it is a global supply chain performance metric, which manages to capture information on how quickly most firms will recover from the impact of disruptions in the worst case. Since the maximum eigenvalue is a measure of how quickly $||\mathbf{p}(t)||$ decays to zero, the norm and hence the resilience do not prioritise the recovery of big firms incurring more losses over small firms incurring small losses. However the resilience will hold in the case where every firm faces an idiosyncratic loss that is constant through its downtime. Suppose a firm $i$ suffers a loss $w_i$ when it is disrupted, then the expected loss for $i$ at time $t$ is $w_i p_i(t)$. It can be shown that the norm vector $\mathbf{w} \circ \mathbf{p}(t)$ follows similar dynamics as $||\mathbf{p}(t)||$. Secondly, the resilience metric could be derived without having the knowledge about the firms disrupted at time $t=0$. In the absence of this knowledge, the set $\mathcal{I}$ is empty and so every node $i$ has a recovery rate equal to the minimum of the recovery rates of its incoming links, $\min_j \mu_{ji}$. Thus the resilience tells us how quickly a supply chain can potentially recover from the effect of disruptions without taking into account where the disruptions originate from. But the resilience metric could be modified to incorporate information on the initially disrupted nodes. If $\mathcal{I}$ is the set of initially disrupted nodes, then the diagonal entries in the matrix $\mathbf{R}_U$ corresponding to $k \in \mathcal{I}$ must be populated by the rates $\mu_k,\ k \in \mathcal{I}$. In this case, the resilience is a measure of how quickly the supply chain can recover given the disruption originates from the nodes in the set $\mathcal{I}$. Thirdly, larger the resilience, better the topology is from the perspective of being able to quickly recover from the effect of disruptions. On the other hand, when the resilience is negative, the exponential convergence enunciated in the previous section does not hold, and it is possible that some disruptions could persist in the network for an arbitrarily long time due to the absence of robust mitigation and contingency strategies for firms in the supply chain.

In addition to the resilience of a supply chain, the relative vulnerability of different firms is a useful metric that could be gleaned from our model. Consider an insurance provider with full knowledge about the topology of the supply chain and the infection and recovery rates. The premium that must be charged from different firms must vary in accordance with the vulnerability arising out of their relative positions in the supply chain. This vulnerability is related to the probability that the firm is disrupted at some point in time. Thus the need of the insurance provider is a positive vector of weights associated with the firms, scaled in accordance with their individual vulnerabilities. We argue in the following lemma that such a weight vector is provided by the principal eigenvector (the eigenvector corresponding to the eigenvalue with the largest real part) of the matrix $\mathbf{R}_U$.

Proposition 1. *Let $\mathbf{u}$ be the eigenvector of $\mathbf{R}_U$ corresponding to the eigenvalue $\lambda_{max}$ with the maximum real part. Then $\lambda_{max}$ is purely real and $\mathbf{u}$ is a strictly positive vector, i.e., $\mathbf{u} > \mathbf{0}$. Also, there exists $t_0$ such that $\forall t > t_0$, $\frac{u_i}{u_j} = \frac{p_i^{SIS}(t)}{p_j^{SIS}(t)}$ for all $i,j$.*

The proof of the lemma is based on eigenspace representation of the solution of a linear dynamical system and is relegated to the appendix. We now consider examples of a few common supply chain topologies in a bid to understand the resilience and relative vulnerability metrics. In order to purely focus on and analyse the effect of topology on resilience, we work in a setting of homogeneous infection and recovery rates in our examples.

*a. Hub and spoke network:* As shown in Figure 2, consider a hub firm (firm 0) which directly procures raw materials or resources from $N$ different firms. Provided these are the only firms in the chain, let $\lambda_{10} = \lambda_{20} = \ldots = \lambda_s$ be the rate at which failure of any of the $N$ firms will disrupt the central hub firm and $\lambda_{01} = \lambda_{02} = \ldots = \lambda_b$ be the rate at which any of the buyer firms get disrupted when the supplier firm is disrupted. The hub firm recovers from its disruptions at rate $\mu_s = \min_j \mu_{j0}$ and the buyer firms recover at rate $\mu_{10} = \mu_{20} = \ldots = \mu_b$. Thus, we have the following rate matrix $\mathbf{R}_U$.

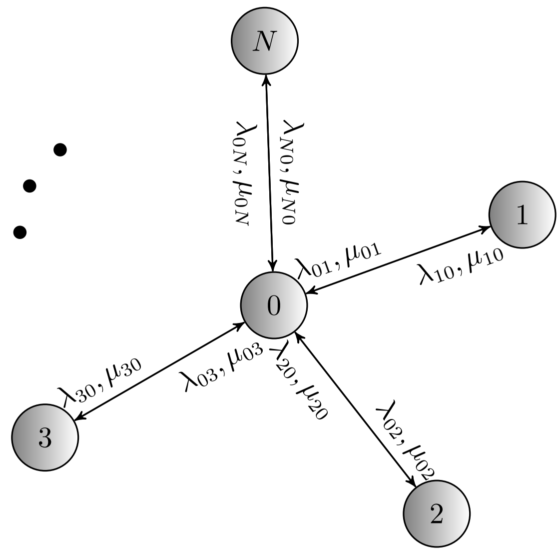


**Figure 2 A hub and spoke supply chain; all spoke firms are suppliers to the hub firm.**

$$
\mathbf{R}_U = \begin{bmatrix} -\mu_s & \lambda_s & \lambda_s & \cdots & \lambda_s \\ \lambda_b & -\mu_b & 0 & \cdots & 0 \\ \lambda_b & 0 & -\mu_b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ \lambda_b & 0 & 0 & \cdots & -\mu_b \end{bmatrix}
$$

The resilience of the hub-and-spoke supply chain, which is the maximum eigenvalue of the matrix, follows from algebraic and matrix computations.

$$
\begin{aligned}
\xi &= -\Re(\lambda_{max}(\mathbf{R}_U)) \\
&= \frac{1}{2}\left(\mu_s + \mu_b - \sqrt{(\mu_s - \mu_b)^2 + 4N\lambda_s\lambda_b}\right)
\end{aligned}
$$

The principal eigenvector corresponding to the maximum eigenvalue representing the relative vulnerabilities is:

$$
\mathbf{u} = \left[\frac{(\mu_b - \mu_s) + \sqrt{4N\lambda_s\lambda_b + (\mu_b - \mu_s)^2}}{2\lambda_b} \; 1 \; 1 \; \ldots \; 1\right]^\top
$$

In the long run, the probability of the central node being disrupted is greater than that of the spoke nodes by a factor of $\frac{(\mu_b - \mu_s) + \sqrt{4N\lambda_s\lambda_b + (\mu_b - \mu_s)^2}}{2\lambda_b}$. When the hub and spoke nodes have identical recovery rates, this factor reduces to $\sqrt{\frac{N\lambda_s}{\lambda_b}}$. Under homogeneous infection rates, the relative vulnerability of the hub node scales as $\sqrt{N}$, the square root of the number of first-tier suppliers to the firm.

*b. Serial supply chain:* Consider a serial supply chain of $N$ firms as shown in Figure 3. Let 0 be a supplier to 1, 1 a supplier to 2 and so on. The supplier-infection rate or the rate at which a firm is disrupted when its supplier is down is $\lambda_s$, and $\lambda_{01} = \lambda_{12} = \lambda_{23} = \ldots = \lambda_s$. The buyer-infection rate or the rate at which a firm is disrupted when its buyer goes down is $\lambda_b$, and $\lambda_{N,N-1} = \ldots = \lambda_{21} = \lambda_{10} = \lambda_b$. Let the contingent recovery rate for the disrupted firm in each of the two cases be $\mu_s$ and $\mu_b$ respectively. However, since we are working with an SIS model, all firms are assumed to have an identical recovery rate of $\mu = \min(\mu_s, \mu_b)$. The rate matrix $\mathbf{R}_U$ is as follows.

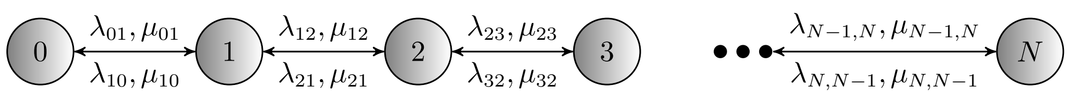


**Figure 3 A serial supply chain where each firm is a supplier to the firm towards its right.**

$$\mathbf{R}_U = \begin{bmatrix} -\mu & \lambda_b & 0 & \cdots & 0 \\ \lambda_s & -\mu & \lambda_b & \cdots & 0 \\ 0 & \lambda_s & -\mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \lambda_s & -\mu \end{bmatrix}$$

With some algebra, the value for the resilience or the maximum eigenvalue of $\mathbf{R}_U$ can be found as shown below.

$$\begin{aligned} \xi &= -\Re(\lambda_{max}(\mathbf{R}_U)) \\ &= \mu - 2\sqrt{\lambda_s \lambda_b} \cos\left(\frac{\pi}{N+1}\right) \end{aligned}$$

The $i^{th}$ component of the eigenvector $\mathbf{u}$, or the relative vulnerability of firm $i$ is $u_i = \left(\frac{\lambda_s}{\lambda_b}\right)^{i/2} \sin\left(\frac{i\pi}{N+1}\right)$. To understand the implications of this eigenvector, we look at three extreme cases:

- $\lambda_s >> \lambda_b$: The dominant factor in the eigenvectors is the $\sqrt{\frac{\lambda_s}{\lambda_b}}$ term. Since the supplier infection proceeds from left to right in Figure 3, the downstream firms have a greater probability of remaining disrupted in the long run. The vulnerability increases by a factor of $k_i\sqrt{\frac{\lambda_s}{\lambda_b}}$ for every firm, as we move downstream from the $i^{th}$ to the $(i+1)^{th}$ firm, where $k_i$ is some nonzero constant based on the sine term.
- $\lambda_s = \lambda_b$: When the infection is equally likely to spread from both suppliers and buyers, the position of the firm in the serial supply chain should be the determining factor for its vulnerability. The sine function from 0 to $\pi$ evaluated at integral multiples of $\frac{\pi}{N+1}$ from $i = 1$ to $i = N$ yields the eigenvector components, $u_i$. This tells us that when all firm are equally likely to be disrupted by their suppliers and buyers, the most central firms have a greater probability of remaining disrupted in the long run compared to the firms in the edges.

- $\lambda_s << \lambda_b$: This case is similar to the first, except for the fact that the upstream firms possess greater vulnerability. As we move from left to right, the vulnerability keeps decreasing by a factor of $\sqrt{\frac{\lambda_s}{\lambda_b}}$ with every firm.

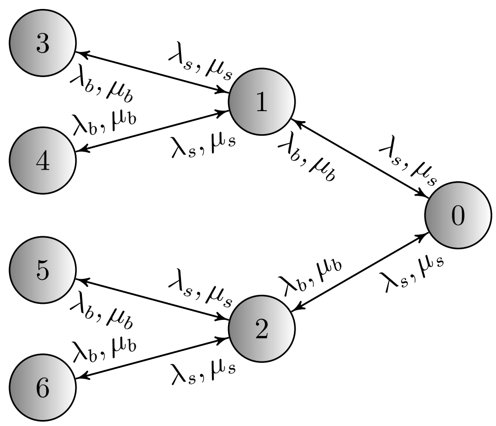


**Figure 4** **An assembly supply chain with four tier-2 suppliers (firms 3,4,5 and 6), two tier-1 suppliers (firms 1 and 2) for firm 0.**

*c. Assembly network:* We consider an assembly network of 7 nodes as in Figure 4. Firms 3 and 4 supply to 1, which in turn supplies to 0. 2 is the second supplier to firm 0, which procures from firms 5 and 6. We again assume that the recovery rates are constant, the supplier infection rate is $\lambda_s$ and the buyer infection rate is $\lambda_b$. As before, we write the rate matrix, resilience and the relative vulnerability:

$$\mathbf{R}_U = \begin{bmatrix} -\mu & \lambda_s & \lambda_s & 0 & 0 & 0 & 0 \\ \lambda_b & -\mu & 0 & \lambda_s & \lambda_s & 0 & 0 \\ \lambda_b & 0 & -\mu & 0 & 0 & \lambda_s & \lambda_s \\ 0 & \lambda_b & 0 & -\mu & 0 & 0 & 0 \\ 0 & \lambda_b & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & \lambda_b & 0 & 0 & -\mu & 0 \\ 0 & 0 & \lambda_b & 0 & 0 & 0 & -\mu \end{bmatrix}$$

$$\begin{aligned} \xi &= -\Re(\lambda_{max}(\mathbf{R}_U)) \\ &= \mu - 2\sqrt{\lambda_s \lambda_b} \\ \mathbf{u} &= \begin{bmatrix} \frac{2\lambda_s}{\lambda_b} & 2\sqrt{\frac{\lambda_s}{\lambda_b}} & 2\sqrt{\frac{\lambda_s}{\lambda_b}} & 1 & 1 & 1 & 1 \end{bmatrix}^\top \end{aligned}$$

*d. Hub node with $M$ suppliers and $N$ buyers:* Consider a hub firm in Figure 5 with $M$ suppliers and $N$ buyers. Assuming homogeneity of supplier and buyer infection rates, the resilience $\xi = \mu - \sqrt{M+N}\sqrt{\lambda_s \lambda_b}$. The relative vulnerability is 1 for the first tier of $M$ suppliers, $\sqrt{(M+N)\frac{\lambda_s}{\lambda_b}}$ for the hub firm and $\frac{\lambda_s}{\lambda_b}$ for the $N$ buyers.

Remark 1. In each of the four examples above, we find that the relative vulnerability of every buyer firm is more than its supplier by a factor $k \times \sqrt{\frac{\lambda_s}{\lambda_b}}$, where $k$ is some constant depending

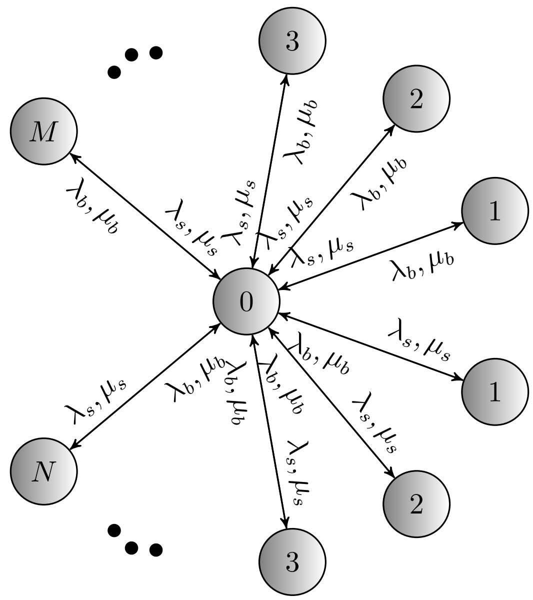


**Figure 5** **A hub firm with $M$ suppliers and $N$ buyers.**

upon the topology and the node's position in the network. When rates are homogeneous across the network, this observation helps us to relate the vulnerabilities of firms in one tier to those in the immediate next.

Remark 2. To compute the relative vulnerabilities, we have used the SIS model which is a model for gleaning the worst case probabilities of disruption for firms. Using the SIS model for resilience makes sense because we have proved in Corollary 1 that the resilience of the SCDC model is upper bounded by the resilience of the SIS model. But such a relationship does not exist between the relative vulnerabilities arising in the SCDC and the SIS models. In other words, if $\mathbf{u}$ is the vector of relative vulnerabilities, it means that after a cutoff time $t_1$, $\frac{\mathbf{p}_i^{SIS}(t)}{\mathbf{p}_j^{SIS}(t)} = \frac{u_i}{u_j}$ (see proof of Proposition 1 in the appendix). However for the SCDC model which is our original model to describe supply chain cascades, it is not necessarily true that $\frac{\mathbf{p}_i^{SCDC}(t)}{\mathbf{p}_j^{SCDC}(t)} = \frac{u_i}{u_j}$. For tractability purposes, we assume that both the resilience and the relative vulnerability metrics are derived for the worst-case model, SIS.

## 6. Conclusion and further work

In this paper, we considered a supply-chain network with a few disrupted firms. We defined metrics for the resilience of the topology and the relative vulnerabilities of different firms to disruptions. We then derived these metrics for some standard supply chain topologies under assumptions of homogeneous rates. Progress needs to be made to understand the tightness of the bounds provided by the SIS model and to get a handle on the relationship between rates, topology and tightness of the bounds. An interesting problem going forward is to compare supply chain topologies. Acemoglu et al. (2013) compare different interbank lending topologies and studies the most and least robust topologies under a specific set of assumptions. However, drawing such a comparison between all possible supply chain topologies among $N$ firms seems absurd at the outset. This is because different supply chain topologies represent different flow of materials and hence may not be comparable to each other. We need to carefully understand what classes of supply chain topologies are comparable with one another to carry out an analysis on the lines of Acemoglu et al. (2013) for supply-chain disruptions. Insights about the relationship between topologies and resilience could be crucial for the problem of endogenizing supply-chain networks.

Another promising avenue is to apply the foregoing analysis on data collected on a firm's internal supply chain or from an insurance provider about a group of firms with a view to identifying the weaknesses in the event of disruptions. It would be a real-world exercise in identifying firms most prone to disruptions and effective risk management strategies in order to improve supply-chain resilience.

While this paper ranked firms in a supply chain in the order of their relative vulnerabilities, a similar ranking can also be undertaken for mitigation and contingency strategies available for firms. This translates to a problem of finding the *links*, improving the rates of which can provide the greatest marginal benefit to the resilience of the supply chain. A sensitivity analysis of the eigenvalues of the matrix $\mathbf{R}_U$ can provide insights into such questions.

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## Appendix A: ODE characterisation for SIS models

We claimed tractability as the reason to bound the SCDC model by an SIS model. In this appendix, we consider an SIS model on a graph with node-specific recovery rates and edge-specific propagation rates and derive ordinary differential equations (ODE) representing the change in probabilities of disruption over time. Consider a graph $G$ and let $\mu_i$ be the recovery rate of node $i$, $\lambda_{ij}$ be the infection rate from $i$ to $j$ and $\mathbf{p}(t)$ be the vector of probabilities of a node being in disrupted state at time $t$. In the following lemma, we present such a system of equations, establish the conditions on an SIS-model for *network recovery* (in other words, to reach the 'all-infection-free' state), and bound the convergence of $||\mathbf{p}(t)||$.

LEMMA 1. *Consider an SIS model on graph $G$ with recovery rates $\mu_i$ and propagation rates $\lambda_{ij}$. Define* $\mathbf{\Lambda} := \begin{bmatrix} 0 & \lambda_{21} & \cdots & \lambda_{n1} \\ \lambda_{12} & 0 & \cdots & \lambda_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1n} & \lambda_{2n} & \cdots & 0 \end{bmatrix}$, $\mathfrak{U} := \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix}$ *and* $\mathbf{R} := \mathbf{\Lambda} - \mathfrak{U}$. *Then,*

1. *The probability vector $\mathbf{p}^{SIS}(t)$ evolves in accordance with the following ODE:*
$$\frac{d\mathbf{p}^{SIS}(t)}{dt} = \mathbf{R}\mathbf{p}^{SIS} - \text{diag}(\mathbf{\Lambda}\mathbf{p}^{SIS})\mathbf{p}^{SIS}$$
2. $\mathbf{p}^{SIS} = 0$ *is an exponentially stable equilibrium if and only if the real parts of all eigenvalues of* $\mathbf{R}$ *are negative.*
3. *When the real parts of all eigenvalues of* $\mathbf{R}$ *are negative, the probabilities* $\mathbf{p}^{SIS}(t)$ *satisfy:*
   (a) $||\mathbf{p}^{SIS}(t)|| \leq .0.$
   (b) $||\mathbf{p}^{SIS}(t)|| \geq ||\mathbf{p}^{SIS}(0)||e^{(\Re(\lambda_{min})+||\mathbf{\Lambda}||_\infty)t}$

*where* $||\mathbf{\Lambda}||_\infty = \max_i \sum_j \lambda_i$ *and* $\Re(\lambda_{max})$ *and* $\Re(\lambda_{min})$ *are the real parts of the eigenvalues of* $\mathbf{R}$ *with the maximum and minimum real parts respectively.*

**Proof of Lemma 1**: **Proof of Part 1**: Every single node in the graph $G$ alternates between states 0 and 1. The time-dependent effective infinitesimal generator matrix $Q_i$ of transitions of a node $i$ is given by:

$$Q_i = \begin{bmatrix} -\sum_j \lambda_{ji} p_j(t) & \sum_j \lambda_{ji} p_j(t) \\ \mu_i & -\mu_i \end{bmatrix}$$

The Kolmogorov forward differential equations corresponding to $Q_i$ are:

$$\begin{bmatrix} \frac{d(1-p_i(t))}{dt} & \frac{dp_i(t)}{dt} \end{bmatrix} = [1 - p_i(t) \quad p_i(t)]\, Q_i$$
$$\frac{dp_i(t)}{dt} = (1 - p_i)\left(\sum_j \lambda_{ji} p_j(t)\right) - p_i \mu_i$$

Combining such an expression for all nodes and writing out in vector form, we get,

$$\frac{d\mathbf{p}(t)}{dt} = \mathbf{R}\mathbf{p} - \text{diag}(\mathbf{\Lambda}\mathbf{p})\mathbf{p} = \mathbf{f}(\mathbf{p})$$

.

**Proof of Part 2**:

The following theorem is a standard result from the theory of differential equations and its proof can be found in Khalil and Grizzle (1996) among others.

Theorem 2. *Let $\mathbf{p}=\mathbf{0}$ be an equilibrium point for the nonlinear system $\frac{d\mathbf{p}(t)}{dt}=f(\mathbf{p})$ where $f(\mathbf{p}):[0,\infty)\to\mathbb{R}^n$ is continuously differentiable, $D=\{\mathbf{p}\in\mathbb{R}^n,\|p\|<r\}$ and the Jacobian matrix $\partial f/\partial p$ is Lipschitz and bounded on $D$, uniformly in $t$. Let $A=\partial f/\partial\mathbf{p}|_{\mathbf{p}=0}$. Then $\mathbf{p}=\mathbf{0}$ is an exponentially stable equilibrium for the nonlinear system if and only if it is an exponentially stable equilibrium for the linear system $\frac{d\mathbf{p}(t)}{dt}=Ap$. Moreover the linear system of equations $\frac{d\mathbf{p}(t)}{dt}=A\mathbf{p}$ converges exponentially to $\mathbf{p}=\mathbf{0}$ if and only if all eigenvalues of $A$ are negative.*

In the ODE system characterising SIS model, $\mathbf{f}(\mathbf{p})=\mathbf{R}\mathbf{p}-\mathrm{diag}(\mathbf{\Lambda}\mathbf{p})\mathbf{p}$ and we need to verify that $\mathbf{f}(\mathbf{p})$ satisfies the Lipschitz-ness and boundedness regularity conditions required in the above theorem. The Jacobian matrix can be computed as:

$$\mathbf{J}=\frac{\partial\mathbf{f}(\mathbf{p})}{\partial\mathbf{p}}=\mathbf{R}-\begin{bmatrix}\sum_j\lambda_{j1}p_j & p_1\lambda_{21} & \cdots & p_1\lambda_{N1}\\ p_2\lambda_{12} & \sum_j\lambda_{j2}p_j & \cdots & p_2\lambda_{N2}\\ \vdots & \vdots & \ddots & \vdots\\ p_N\lambda_{1N} & p_N\lambda_{2N} & \cdots & \sum_j\lambda_{jN}p_j\end{bmatrix}$$

Lemma 2. *The Jacobian matrix $\mathbf{J}$ is bounded and Lipschitz on a set $D=\{\mathbf{p}\in\mathbb{R}^n,\|\mathbf{p}\|<1\}$, uniformly in $t$.*

*Proof*: To prove the boundedness of $\mathbf{J}$, i.e. $||\mathbf{J}||\le K$ for some constant $K$, we consider the infinity-norm of a matrix, defined as $||\mathbf{J}||_\infty=\max_i\sum_j|\mathbf{J}_{ij}|$. Since $\lambda$s and $p$ are bounded, all absolute row sums are bounded, and so is the norm $||\mathbf{J}||_\infty$.

The Lipschitz boundedness can be identified by evaluating $\mathbf{J}$ at $\mathbf{p}_1$ and $\mathbf{p}_2$. We show $||\mathbf{J}_1-\mathbf{J}_2||_\infty<||\mathbf{p}_1-\mathbf{p}_2||_\infty$.

$$\begin{aligned}||\mathbf{J}_1-\mathbf{J}_2||_\infty&=\max_i\sum_j\lambda_{ji}|p_j^1-p_j^2|+\left(\sum_{j=1,j\neq i}^{N}\lambda_{ji}\right)|p_i^1-p_i^2|\\&=\lambda_{1i^*}|p_1^1-p_1^2|+\lambda_{2i^*}|p_2^1-p_2^2|+\ldots+\lambda_{Ni^*}|p_N^1-p_N^2|+|p_{i^*}^1-p_{i^*}^2|\left(\sum_{j=1,j\neq i^*}^{N}\lambda_{ji^*}\right)\\&\le\left[\lambda_{1i^*}+\lambda_{2i^*}+\ldots+\lambda_{Ni^*}+\sum_{j=1,j\neq i^*}^{N}\lambda_{ji^*}\right]\max_j\,|p_j^1-p_j^2|\\&\le M||\mathbf{p}_1-\mathbf{p}_2||_\infty\end{aligned}$$

□

Since the Jacobian $\mathbf{J}$ evaluated at $\mathbf{p}=0$, $\mathbf{J}|_{\mathbf{p}=0}=R$, Theorem 2 tells us that $\mathbf{p}=0$ is an exponentially stable equilibrium for the nonlinear system $\frac{d\mathbf{p}(t)}{dt}=\mathbf{f}(\mathbf{p})$ if and only if it is an exponentially stable equilibrium for the system $\frac{d\mathbf{p}(t)}{dt}=R\mathbf{p}$. Moreover, for $\mathbf{p}=0$ to be an exponentially stable equilibrium for $\frac{d\mathbf{p}(t)}{dt}=R\mathbf{p}$, the real parts of all eigenvalues of $R$ must be negative. Hence $\mathbf{p}$ converges exponentially to 0 if and only if the real parts of all eigenvalues of $R$ are negative.

**Proof of Part 3**:

$$\begin{aligned}\frac{d||\mathbf{p}(t)||^2}{dt}&=\left(\frac{d\mathbf{p}(t)}{dt}\right)^\top\mathbf{p}(t)+\mathbf{p}(t)^\top\frac{d\mathbf{p}(t)}{dt}\\&=(\mathbf{R}\mathbf{p}-\mathrm{diag}(\mathbf{\Lambda}\mathbf{p})\mathbf{p})^\top\mathbf{p}+\mathbf{p}^\top(\mathbf{R}\mathbf{p}-\mathrm{diag}(\mathbf{\Lambda}\mathbf{p})\mathbf{p})\\&=\mathbf{p}^\top(\mathbf{R}+\mathbf{R}^\top)\mathbf{p}-2\mathbf{p}^\top(\mathrm{diag}(\mathbf{\Lambda}\mathbf{p})\mathbf{p})\end{aligned}$$

$$\leq \mathbf{p}^\top(\mathbf{R}+\mathbf{R}^\top)\mathbf{p}$$
$$\leq 2\Re(\lambda_{max})||\mathbf{p}||^2$$

The first inequality is because the vectors $\mathbf{p}$ and $\mathbf{p}\circ\mathbf{\Lambda p}$ are made of positive elements. The second is obtained by noting $\mathbf{R}+\mathbf{R}^\top$ is a symmetric matrix and expanding it as a quadratic form expressed through its eigenvalues. Therefore we deduce the following:

$$||\mathbf{p}(t)|| \leq ||\mathbf{p}(0)||e^{\Re(\lambda_{max})t}$$

In a similar vein, we obtain the following lower bounds.

$$\begin{aligned}
\frac{d||\mathbf{p}(t)||^2}{dt} &= \mathbf{p}^\top(\mathbf{R}+\mathbf{R}^\top)\mathbf{p} - 2\mathbf{p}^\top(\text{diag}(\mathbf{\Lambda p})\mathbf{p}) \\
&\geq 2\Re(\lambda_{min})||\mathbf{p}||^2 - 2||\mathbf{p}||||\text{diag}(\mathbf{\Lambda p})\mathbf{p}|| \\
&\geq 2(\Re(\lambda_{min}) - ||\mathbf{\Lambda}||_\infty)||\mathbf{p}||^2 \\
\implies ||\mathbf{p}(t)|| &\geq ||\mathbf{p}(0)||e^{(\Re(\lambda_{min})-||\mathbf{\Lambda}||_\infty)t}
\end{aligned}$$

The first inequality is Cauchy-Schwartz; for the second, note that $||\text{diag}(\mathbf{\Lambda p})\mathbf{p}||^2 = \sum_j \left[p_j\left(\sum_i \lambda_{ji}p_i\right)\right]^2 \leq \sum_j \left[p_j\left(\sum_i \lambda_{ji}\right)\right]^2 \leq ||\mathbf{\Lambda}||_\infty^2 \sum_j p_j^2$.

## Appendix B: Proofs

**Proof of Theorem 1** We use a stochastic coupling argument to prove this result. The key lemma is the following.

LEMMA 3. *Suppose $X(t)$ and $Y(t)$ are two continuous time Markov chains starting at time $t=0$ and $X(0)=Y(0)=0$. The processes obey the following transition rates:*

$$\begin{aligned} X(t): & \begin{cases} 0 \to 1, \textit{at rate } A \\ 1 \to 0, \textit{at rate } B \end{cases} \\ Y(t): & \begin{cases} 0 \to 1, \textit{at rate } C \\ 1 \to 0, \textit{at rate } D \end{cases} \end{aligned} \tag{2}$$

*If $A \geq C$ and $B \leq D$, then $Y(t) \leq X(t)$ almost surely for all $t$ and so, $P(X(t)=1) \geq P(Y(t)=1)$, $\forall\ t$.*

**Proof of Lemma 3**: The lemma can be proved using standard coupling argument. Define $Z_t = (X_t, Y_t)$ with the following transition rates.

- From $Z_t = (0,0)$, it is possible to transition independently to $(1,1)$ at rate $C$ and to $(1,0)$ at rate $A-C$
- From $Z_t = (1,1)$, it is possible to transition independently to $(0,0)$ at rate $B$ and to $(1,0)$ at rate $D-B$
- From $Z_t = (1,0)$, it is possible to transition independently to $(0,0)$ at rate $B$ and to $(1,1)$ at rate $C$

Under this definition, the processes $X_t$ and $Y_t$ have the desired transition rates posed in the lemma. We note that starting from the state $Z_0 = (0,0)$, one never visits the state $(0,1)$, and the only allowable states are $(0,0)$, $(1,1)$ and $(1,0)$. In other words, $Y_t \leqslant X_t$ almost surely for every $t$. In particular, $[Y_t = 1] \subseteq [X_t = 1]$ hence $\mathbb{P}(Y_t = 1) \leqslant \mathbb{P}(X_t = 1)$. Q.E.D.

Thus the above lemma provides a way to relate probabilities of firms being disrupted in different CTMCs defined on the same state space, wherein the different CTMCs could correspond to transition rates specified by different models. We now show that the $0 \to 1$ transition rates for nodes are the same in the SCDC and SIS models, but the $1 \to 0$ transition rates in SCDC model is lower bounded by the corresponding rates in SIS model.

**Observation** 3. Under SCDC model, for every node $i \in V \setminus \mathcal{I}$, $|j : e_{ji}(t) = 1| \leq 1\ \forall\ t \geq 0$. Moreover, for all $t$ when $v_i(t) = 1$, $|j : e_{ji}(t) = 1| = 1$.

To see this, note that at any instant in time a firm $i \notin \mathcal{I}$ can be in one of three states: undisrupted and with undisrupted neighbours, undisrupted, but with one or more disrupted neighbours and disrupted. In the first two cases, $|j : e_{ji}(t) = 1| = 0$, because if an incoming edge is disrupted, a firm has to be in disrupted state. In the third case when the firm is in disrupted state, $|j : e_{ji}(t) = 1| = 1$, because when a firm is disrupted, it cannot be disrupted by its neighbours unless it recovers. Thus when evolving under the SCDC model, at any time, a disrupted node has strictly one incoming edge in state 1.

**Observation** 4. The $0 \to 1$ transition rate is equal in the SCDC and SIS models; the $1 \to 0$ transition rate in SCDC model is lower bounded by the corresponding rate in SIS model.

That the transition rate of $1 \to 0$ for nodes $i \notin \mathcal{I}$ is lesser in SIS than in SCDC can be understood from the following set of inequalities. For $i \notin \mathcal{I}$,

$$\begin{aligned}\sum_j \mathbb{1}_{e_{ji}=1}(\mu_{ji}+R_j) &\geq \min_j \mathbb{1}_{e_{ji}=1}(\mu_{ji}+R_j), \quad \text{from Observation 3,}\\ &\geq \min_j \mathbb{1}_{e_{ji}=1}\mu_{ji}, \quad \text{since} R_j \geq 0,\\ &\geq \min_j \mu_{ji}.\end{aligned}$$

The first inequality follows from Observation 3, as every infected node $i \notin \mathcal{I}$ has exactly one incoming edge with $e_{ji}=1$. Since the summation is non-zero for a single value of $j$, it is greater than the minimum over all $j$. The last inequality holds, as under SIS, $\mathbb{1}_{e_{ji}=1}=1$ for $i \notin \mathcal{I}$ when $v_i = 1$.

Applying the result from Observations 3 and 4 to Lemma 3, we conclude $\mathbb{P}(X_i^{SIS}(t)=1) \geq \mathbb{P}(X_i^{SCDC}(t)=1)$ for all $i$ and $t>0$ and consequently, $||\mathbf{p}^{SCDC}(t)|| \leq ||\mathbf{p}^{SIS}(t)||$. Hence Theorem 1 is proved.

**Proof of Corollary 1**: Follows from Lemma 1 and is omitted.

**Proof of Proposition 1**:

From Lemma 1, the Kolmogorov differential equation system for SIS model is:

$$\frac{d\mathbf{p}^{SIS}(t)}{dt} = \mathbf{R}_U \mathbf{p}^{SIS} - \text{diag}(\mathbf{\Lambda}\mathbf{p}^{SIS})\mathbf{p}^{SIS}$$

From the theory of nonlinear differential equations, we understand that this system behaves like its linearised version when close to the equilibrium of $\mathbf{p}^{SIS} = \mathbf{0}$. Since the Jacobian of $\mathbf{f}(\mathbf{p}) = \mathbf{R}_U\mathbf{p}^{SIS} - \text{diag}(\mathbf{\Lambda}\mathbf{p}^{SIS})\mathbf{p}^{SIS}$ is $\mathbf{R}_U$, this linearised system is $\frac{d\mathbf{p}^{SIS}(t)}{dt} = \mathbf{R}_U\mathbf{p}^{SIS}(t)$. In other words, $\exists\, t_0$ such that $\forall t \geq t_0$, $\frac{d\mathbf{p}^{SIS}}{dt} = \mathbf{R}_U\mathbf{p}^{SIS}$. Since $\mathbf{R}_U$ is a quasipositive matrix (matrix with all off-diagonal entries non-negative), by properties of quasipositive matrices, $\mathbf{R}_U$ has a unique, real and negative maximum eigenvalue $\lambda_1$ and a positive eigenvector $\mathbf{u}_1$ corresponding to $\lambda_1$. Let the eigenvalues of $\mathbf{R}_U$ be $\lambda_i$ and $\mathbf{u}_i$ for $i = 1,2,3,\ldots,N$. The eigenspace representation of the solution of the linearised system is:

$$\mathbf{p}^{SIS}(t) = \sum_i c_i e^{\lambda_i t}\mathbf{u}_i$$

Since $\lambda_1$ is the maximum eigenvalue and using limiting property of the exponential, there exists $t_1 \geq t_0$ such that $\forall\ t \geq t_1$, $\mathbf{p}^{SIS}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1$. Since $\mathbf{u}_1$ is positive, $\frac{\mathbf{p}_i^{SIS}(t)}{\mathbf{p}_j^{SIS}(t)} = \frac{u_i}{u_j}$, for all $t \geq t_1$ and for all $i,j = 1,2,\ldots,N$.